

Superfluid nucleon matter in and out of equilibrium and weak interactions*

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The Larkin-Migdal approach to a cold superfluid Fermi liquid is generalized for a non-equilibrium system. The Schwinger-Keldysh diagram technique is applied. The developed formalism is applicable to the pairing in the states with arbitrary angular momenta. We consider the white body radiation problem by calculating probabilities of different direct reactions from a piece of a fermion superfluid. The closed diagram technique is formulated in terms of the full Green's functions for systems with the pairing correlation. The cutting rules are used to classify the diagrams representing one-nucleon, two-nucleon, etc. processes in the matter. The important role of multi-piece diagrams for the vector-current conservation is demonstrated. In the case of equilibrated systems, dealing with dressed Green's functions, we demonstrate correspondence between calculations in the Schwinger-Kadanoff-Baym-Keldysh formalism and the ordinary Matsubara technique. As an example we consider neutrino radiation from the neutron pair breaking and formation processes in case of a singlet pairing. Necessary correlation effects are included. The in-medium renormalization of normal and anomalous vertices is performed.

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I. INTRODUCTION

A. Historical remarks

The phenomenological theory of normal Fermi liquids at zero temperature was proposed by L.D. Landau in Refs. [1, 2]. A.B. Migdal made the very important observation that a jump in the particle momentum distribution at the Fermi momentum corresponds to a pole of the fermion Green's function in the normal Fermi liquid and is preserved even in the strongly interacting system [3]. The presence of the pole contribution to the fermion Green's function allowed Galitsky and Migdal to develop the Green's function formalism for many-body fermionic systems, see Ref. [4]. These concepts were first elaborated on example of low-lying particle-hole excitations in Fermi liquids. A.B. Migdal was first who applied these methods to description of various nuclear phenomena and constructed a closed semi-microscopic approach that is usually called "Theory of finite Fermi systems" [5, 6].

A general understanding of the phenomenon of superconductivity in case of the weak attraction between fermions was achieved by J. Bardeen, L.N. Cooper and J.R. Schrieffer in Ref. [7], see also Ref. [8] for detailed exposition. Due to the sharpness of the Fermi surface provided by presence of the Migdal's jump one can consider the fermions on the Fermi surface as moving in an effective two-dimensional momentum space. It follows immediately that even a weak attraction between two particles is sufficient to form a Cooper pair. As soon as the pairing phenomenon is established one can follow different routes in description of the superconductivity and superfluidity: In Ref. [9] N.N. Bogolyubov suggested a very convenient transformation from the particle ψ operators to the new operators of effective excitations on top of the background of Cooper pairs. This transformation is broadly used in the theory of superconductivity. L.P. Gorkov developed the Green's function formalism for superconducting fermion systems with an electron-phonon interaction [10]. Y. Nambu introduced a matrix formalism to the theory of superconducting metals [11] (Green's functions formulated in the Nambu-Gorkov space). In Ref. [12] G.M. Eliashberg extended the A.B. Migdal's theory of the strong electron-phonon interaction in normal metals [13] to include the Cooper pairs. This approach can be used to describe strong coupling superconductors. Also A.B. Migdal was the first who rose the idea about a possibility of the neutron-neutron pairing and superfluidity in neutron stars, which were hypothetical objects that time [14].

The Fermi liquid theory was then generalized by A.I. Larkin and A.B. Migdal for the description of fermion superfluids at zero temperature [15]. Their formulation is more general than that done in papers by Y. Nambu and L.P. Gorkov, since it allows for differ-

ent interactions in the particle-particle and particle-hole channels. A.J. Leggett applied this formalism for the superfluid ^3He at a finite temperature [16, 17]. J. Schwinger in Ref. [18], L.P. Kadanoff and G. Baym in Ref. [19], and L.P. Keldysh in Ref. [20] developed the non-equilibrium diagram technique for the description of non-equilibrium Fermi and Bose systems. Even for equilibrium systems at $T \neq 0$ Schwinger-Kadanoff-Baym-Keldysh approach is in many cases more convenient than the standard Matsubara technique (applicable only for equilibrium systems) since it does not involve the Wick rotation and the obtained results can be continuously transformed to those computed in the standard Feynman-diagram technique at zero temperature.

The importance of coherence time effects on the production and absorption of field quanta from the motion of source particles in matter has been first discussed by L.D. Landau and I.Ya. Pomeranchuk [21]. In Ref. [22] A.B. Migdal developed the complete theoretical framework for the description of the bremsstrahlung radiation of ultra-relativistic electrons in the process of multiple rescatterings on Coulomb centers. Successful measurements of such a suppression of the bremsstrahlung radiation have been recently carried out at the Stanford Linear Accelerator Center [23], see also the review in Ref. [24]. Now this effect is named the Landau-Pomeranchuk-Migdal effect.

In the framework of his theory of finite Fermi systems A.B. Migdal developed the description of the soft pion degree of freedom in nuclear matter in application to atomic nuclei and neutron stars. In vacuum, pions are the lightest quanta of the strong interactions between baryons. In medium, pionic modes are softened even further due to the coupling to nucleon particle-hole modes and can be easily excited even at low excitation energies, similar to phonons in solids. As an intriguing consequence of the pion softening A.B. Migdal suggested a possibility of the pion condensation¹ at the increase of the baryon density, see Refs. [25, 27, 28]. Latter on, in analogy to the pion condensation, the ideas of the kaon condensation [29] and the charged ρ meson condensation [30, 31] in the interiors of neutron stars were explored. Softening of the pionic mode at finite temperature [32–34] and at non-equilibrium [35–37] may manifest in neutron stars [38] and heavy ion collisions [36, 39].

A.B. Migdal rose question on a possibility of existence of superdense abnormal nuclei glued by the pion condensate [25, 40]. Also a possibility of nuclei-stars was considered in Ref. [41]. The similar ideas on a possibility of quark nuclei, quark stars and hybrid stars [42, 43] are continued to be extensively explored nowadays, see [44, 45].

¹ Independently pion condensation was also suggested by D.J. Scalapino and R. Sawyer [26].

B. White body radiation

Below we consider the white-body radiation from a piece of a superfluid fermion matter. To be specific we focus on the neutrino radiation from a piece of superfluid nucleon matter. Standard Feynman technique of summation of squared matrix elements of reactions fails to calculate reaction rates in the medium, since in general case there are no asymptotic states for source particles in matter. Indeed, source particles continue to collide before and after radiation of a quantum. This gives rise to finite imaginary parts of the self-energy functions (particle widths). If one naively replaced the summation of all perturbative Feynman diagrams (with free Green's functions) by the summation of corresponding diagrams with dressed Green's functions, it would lead to a double counting due to multiple repetitions of some processes (for an extensive discussion of how one can treat this defect see [35, 46–48]). This calls for a formalism dealing with closed diagrams (integrated over all possible in-medium particle states) with full non-equilibrium Green's functions. Such a general formalism was developed in Ref. [47]. It treats on equal footing one-fermion and multi-fermion processes as well as resonance reaction contributions of the boson origin, such as processes with participation of zero sounds and reactions on the boson condensates. Decomposition of diagrams is done in terms of the full G^{-+} Green's functions (Wigner densities). Each diagram in the series with full Green's functions is free from the infrared divergencies. In such a way one generalizes Landau-Pomeranchuk-Migdal treatment of the multiple scattering on external centers to the treatment of the multiple scattering in matter. Both, the correct quasi-particle and quasi-classical limits are recovered.

The formulation of the radiation problem in terms of closed diagrams calculated within the non-equilibrium Green's function in quasi-particle approximation was performed in Ref. [35]. This approach was called the "*optical theorem formalism*". In Refs. [35, 46] it was demonstrated that the standard calculations of reaction rates via integration of squared reaction matrix elements and the results of the optical theorem formalism match exactly, provided conditions for the quasi-particle approximation for fermions are fulfilled. Formally the matching is done by cutting the closed diagrams. In general case considered in Ref. [47] going beyond the quasi-particle approximation, the series of closed diagrams is constructed with respect to the number of the G^{-+} Green's functions. For low temperatures each G^{-+} line brings extra $(T/\epsilon_F)^2$ factor in the production rate of the radiating quanta, ϵ_F is the Fermi energy. In Ref. [47] the relations between reaction rates at finite widths and the quasi-particle rates were found.

All real calculations of fermion superfluids were performed within quasi-particle approximation for fermions (when fermion width is much less than all other relevant energetic scales). Below we focus on the Larkin-Migdal

approach to the cold fermion superfluids and formulate it in terms of the Schwinger-Kadanoff-Baym-Keldysh technique to describe fermion superfluids in equilibrium at $T \neq 0$ and out of equilibrium.

C. Neutrino cooling of neutron stars

Physics of neutron star cooling is based on a number of ingredients, among which the neutrino emissivity of the high-density hadronic matter in the star core is the important one.

After the first tens of seconds (at most hours), the typical temperature of a neutron star decreases below the so-called neutrino-opacity temperature $T_{\text{opac}} \sim (1-2)$ MeV. At these conditions neutrinos and anti-neutrinos can be radiated directly from the star interiors without subsequent rescattering, since their mean-free path is much longer than the star radius [46]. Hence, the star can be considered as a piece of a warm "white" body for neutrinos. Typical averaged lepton energy (\sim several T) is much larger than the nucleon particle width $\Gamma_N \sim T^2/\epsilon_F$. Therefore, the nucleons can be treated within the quasi-particle approximation. This observation simplifies consideration essentially. One usually follows an intuitive way for the separation of the processes according to their phase spaces. The one-nucleon processes (if they are not forbidden by the energy-momentum conservation) have the largest emissivity, $\epsilon_\nu \propto T^6$ for non-superfluid systems, then two-nucleon processes come into play, $\epsilon_\nu \propto T^8$, and so on. In the optical theorem formalism one-nucleon processes are determined by the self-energy Σ^{-+} of virtual W and Z bosons expanded in the series with respect to the number, N , of $G^{-+}G^{+-}$ loops with full " $++$ " and " $--$ " vertices. The $N = 1$ diagrams correspond to one-nucleon processes, the $N = 2$ diagrams to the two-nucleon processes, etc.

In the so-called "*standard scenario*" of the neutron star cooling, the processes were calculated without taking into account in-medium effects. It was argued that the most important channel at temperatures up to $T \sim 10^8-10^9$ K is the modified Urca (MU) process $nn \rightarrow npe\bar{\nu}$. First estimates of the MU emissivity were done in [49, 50]. In Ref. [51, 52] B. Friman and O.V. Maxwell recalculated the emissivity of this process in the model, where the nucleon-nucleon (NN) interaction was approximated by a free one-pion exchange (FOPE). The expression of the neutrino emissivity obtained by them was used in various computer simulations, e.g., in Refs. [53–55]. Besides the MU process, the "*standard scenario*" includes also the processes of the nucleon (neutron and proton) bremsstrahlung (NB) $nn \rightarrow nn\nu\bar{\nu}$ and $np \rightarrow np\nu\bar{\nu}$, which contributions to the emissivity is smaller than those of the MU processes, see Refs. [51, 56]. The density dependence of the reaction rates calculated with the FOPE is rather weak and the neutrino radiation from a neutron star depends very weakly on the star mass.

There exists another class of so-called "exotic" pro-

cesses, which occur only if some special condition is fulfilled, i.e. when the nucleon density exceeds some critical values. These are the direct Urca (DU) processes on nucleons (e.g., $n \rightarrow pe\bar{\nu}$) and hyperons [57], pion Urca reactions on a pion condensate [46, 58, 59], kaon Urca processes on a kaon condensate [60, 61], ρ -Urca processes on charged ρ -condensates [30, 31], DU processes on quarks [62], DU processes on fermion condensates [63]. The values of critical densities are different for various processes and are model dependent. For example, some relativistic mean-field models produce the critical density of the DU reaction, n_c^{DU} as low as the nuclear saturation density $n_0 \simeq 0.16 \text{ fm}^{-3}$. However, the realistic, microscopically-based Urbana-Argonne equation of state [64] yields $n_c^{\text{DU}} \simeq 5n_0$. The simulations of the neutron star cooling history in Refs. [65, 66] have shown that the occurrence of the DU processes in the neutron star with masses $M < 1.5M_\odot$ would lead to problems with the explanation of soft X -ray data. The constraint on the equation of state of the dense nuclear matter, requiring a sufficiently high value of n_c^{DU} , was proposed in Ref. [31] and explored in details in Ref. [67].

It was shown in Refs. [28, 35, 38, 46, 59, 68] that the neutrino emission from dense hadronic component in neutron stars is subject of strong modifications due to collective effects in the nuclear matter. Many new reaction channels open up in medium in comparison to the vacuum. In Refs. [28, 35, 38, 46, 59, 68] the nucleon-nucleon interaction was considered within the Landau-Migdal approach to Fermi liquids. The softening of the in-medium one-pion exchange (MOPE) mode and other medium polarization effects, like nucleon-nucleon correlations in the vertices, renormalization of the local part of NN interaction due to loop effects, as well as a possibility of the neutrino emission from intermediate reaction states and DU-like reactions involving zero sounds and boson condensates were incorporated. It was demonstrated in Refs. [28, 38, 46, 68], that for $n \gtrsim n_0$ the neutrino emissivity is mainly determined by the medium modified Urca (MMU) process, in which the neutrino is radiated from the intermediate reaction states. This fact changes significantly the absolute value and the density dependence of the $nn \rightarrow npe\bar{\nu}$ process rate. The latter becomes very strong. Therefore, for neutron stars with larger masses the resulting emissivity of the MMU process proves to be substantially higher than the corresponding value (MU) calculated in the FOPE model of Ref. [51]. For $n \gtrsim n_0$, the medium-modified nucleon bremsstrahlung (MNB) processes yield a smaller contribution than MMU ones since the former does not include the neutrino radiation from intermediate states, e.g. from intermediate pion. However, the MNB processes are more efficient than the NB ones for such densities.

Oppositely, for $n < n_0$ the in-medium effects can moderately suppress the two-nucleon reaction rates compared with those given by the FOPE model [69–71]. The pion softening effect disappears at $n \lesssim 0.5\text{--}0.7n_0$, see

Refs. [28, 38], but the nucleon-nucleon short-range repulsion effect remains. Inclusion of the nucleon-nucleon correlations without the pion softening [72] yields a suppression effect also at $n \gtrsim n_0$. Obviously, this effect also follows from general consideration in Refs. [28, 46], if one artificially suppresses the pion softening effect.

After the seminal work of A.B. Migdal [14], various aspects of the nucleon superfluidity in neutron stars were studied in the literature: The presence of a nucleon superfluid interacting with the normal component is needed for explanation of glitches in pulsar periods and neutron star quakes [73]. Explanation of pulsar cooling curves also requires an inclusion of superfluid phases [66]. Several superfluid phases are found possible. Phase transitions between different phases may take place [74].

It is commonly accepted that most important are the superfluid phases with the spin-singlet pairing of neutrons and protons, in the $1S_0$ state, and the spin-triplet pairing of neutrons in the $3P_2$ states. The latter is believed to occur in neutron star interior at $n \gtrsim n_0$ in the state with $m_J = 0$, where m_J is the projection of the total pair momentum onto a quantization axis. In case $|m_J| = 2$ the exponential suppression of the specific heat and the neutrino emissivity is replaced by a power-law suppression since the gap vanishes at the poles of the Fermi sphere. This possibility was mentioned for the first time in Ref. [35], the corresponding reaction rates were calculated in Ref. [75]. However, a mechanism to realize this interesting possibility in neutron star cooling was not elaborated yet.

Many papers are devoted to the calculation of pairing gaps within different approaches [76–87]. The obtained results can be essentially different depending on a model for the nucleon-nucleon interaction and a calculation scheme. The predictions of the neutron $3P_2$ gaps are especially uncertain, e.g., compare Refs. [84] and [85]. For review see Ref. [88] and references therein. Ref. [84] argues that $3P_2$ gap should be strongly suppressed whereas Ref. [85] argues for its strong enhancement. Reference [66] calculated cooling curves using both these assumptions and concluded that the cooling history is naturally explained within assumption on the suppressed $3P_2$ gap. Recently Ref. [89] studied the new data on the cooling of Cas A object. Their conclusion is in favor of a suppressed $3P_2$ gap. At temperatures below the critical temperatures of the neutron, T_{cn} , and proton, T_{cp} , pairing, the reaction rates, considered above, are suppressed because of a decrease of the available phase space. Initially, the suppression effects were included simply by multiplying the rate of a two-nucleon process by the factors $e^{-2\Delta/T}$ [52]. Later, the phase-space suppression factors (so called R factors) have been treated more accurately in Ref. [90].

In nucleon superfluids, there exist new neutrino-production mechanisms, which are forbidden for $T > T_c$. These are the processes, suggested in Ref. [35, 68, 91], in which the creation of a neutrino–anti-neutrino pair is associated with the breaking and formation of a Cooper

pair – the so-called nucleon pair breaking and formation (PBF) processes. The emissivities of the nucleon PBF processes are suppressed at $T < T_c$ by the same factor $\sim \exp(-2\Delta/T)$ as for the MU, NB, MMU and MNB processes. However, in comparison to the all latter processes, the nucleon PBF processes have the large one-nucleon phase-space volume [35, 68]. The existence of this new cooling mechanism demonstrates that influence of the nucleon pairing on the neutrino production rates cannot be reduced just to an introduction of a simple phase-space suppression factor.

Early works [35, 38, 68, 75, 90–92] which studied the PBF processes, did not care about the conservation of the weak vector current. The latter is fulfilled only if the in-medium renormalization of weak vertex functions is performed in accord with the renormalization of Green's functions. This problem was tackled in Refs. [93–99]. Reference [93] argued that the emissivity of the $1S_0$ PBF processes should be dramatically suppressed as $\propto v_F^4$, where v_F is the Fermi velocity of non-relativistic nucleons, provided the vector current conservation constraint is taken into account. The consistent calculation of the PBF emissivity induced by the vector and axial-vector currents was performed in Ref. [94, 95] within the Larkin-Migdal-Leggett Fermi-liquid approach. The latter takes properly into account correlation effects in both particle-particle and particle-hole channels. It was demonstrated that the neutrino emissivity is actually controlled by the axial-vector current and is suppressed only by the factor $\propto v_F^2$, rather than $\propto v_F^4$. Both neutron PBF and proton PBF processes yield contributions of the same order of magnitude provided strong and electromagnetic renormalizations of the proton weak vertices [35, 100, 101] are included. In Ref. [96] one argues that for the $3P_2$ neutron pairing the vector current conservation changes moderately the result obtained without its inclusion. As pointed out in Ref. [102], the suppression of the PBF processes at low densities might be served as a possible explanation of the superburst ignition.

As we have mentioned, the convenient Nambu-Gorkov formalism developed for the description of metallic superconductors, cf. Refs. [8, 10, 11], does not distinguish interactions in particle-particle and particle-hole channels. These interactions can be, however, essentially different in a strongly interacting system, like in the nuclear matter and in the liquid He^3 . The adequate methods for Fermi liquids with pairing were developed for zero temperature by A.I. Larkin and A.B Migdal in Ref. [15] (see also [6]) and for a finite temperature by A.J. Leggett in Ref. [16, 17]. The problem of calculation of a response function of a Fermi system to an external interaction becomes tractable at cost of introduction of a set of Landau-Migdal parameters for quasi-particle interactions. Parameters can be either evaluated microscopically or extracted from analysis of experimental data, see Ref. [6]. The technical difference of the Larkin-Migdal and Leggett approaches is that the former approach works out equations for full in-medium ver-

tices, whereas the latter one calculates directly a response function. The former approach was aimed at the study of transitions in nuclei, and the latter on the analyzes of collective modes in superfluid Fermi liquid. The principal equivalence of both approaches was emphasized already by A.J. Leggett in Refs. [16, 17]. Reference [95] demonstrates how one may use both approaches in calculations of the PBF rates.

Reference [103] was the first one, in which the most important in-medium effects were incorporated in the numerical code for neutron star cooling. Among them neutron PBF and proton PBF processes were treated as equally important. The PBF processes (but with free vertices) were incorporated also in the “standard” cooling scenario [90, 104] that led the authors of Ref. [105] to the suggestion of the minimal cooling paradigm. Detailed simulations of different medium effects have been done in [65, 66]. In contrast to the minimal cooling paradigm, the medium modifications of all reaction rates lead to their pronounced density dependence. For the PBF processes it is mainly due to the dependence of pairing gaps and nucleon-nucleon correlation factors on the density. For MMU processes the reaction matrix elements are strongly density dependent due to the softening of the exchanged pion and the dependence of nucleon-nucleon correlation factors on the density. It establishes the strong link between the cooling behavior of a neutron star and its mass [28, 46, 59, 65, 66, 103]. The density dependence of the reaction rates provides a smooth transition from “standard” to “non-standard” cooling for the increasing star-center density, i.e., for increasing the star mass. Thus, the inclusion of the most important in-medium effects within the “*nuclear medium cooling scenario*” enables us to describe appropriately both high and low surface temperatures obtained from analyzes of soft X-ray pulsar data. The mentioned above moderate suppression of the PBF emissivity ($\propto v_F^2$) at $1S_0$ pairing should not significantly affect general conclusions on the neutron star cooling history done in previous works where it was not incorporated.

The paper is organized as follows. In Section II we formulate description of normal Fermi liquids at non-equilibrium. Softening of pionic degrees of freedom is taken into account. In Section III we perform generalizations to the fermion superfluids at non-equilibrium. The Larkin-Migdal equations are formulated on the Schwinger-Keldysh contour. A possibility of the pairing in an arbitrary momentum state is considered. In Section IV we introduce optical theorem formalism for normal and superfluid fermion systems out of equilibrium. Cutting rules for closed diagrams expanded in series with respect to the number of G^{--} full Green's functions are formulated. Important role of multi-piece diagrams is shown. Fermi liquid renormalizations are performed in Section V. Equilibrium $T \neq 0$ systems with pairing are considered in Section VI. In Section VII, as an example, we find the current-current correlator and in Section VIII, the neutrino emissivity from the PBF processes on

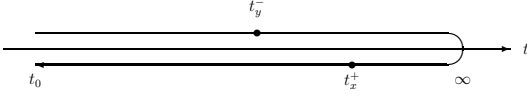


FIG. 1: Closed real-time contour with two external points x, y on the contour

neutrons paired in $1S_0$ state. Technical details are given in Appendices.

II. DESCRIPTION OF NON-EQUILIBRIUM NORMAL FERMI LIQUIDS

A. Dyson equations on Schwinger-Keldysh contour

The non-equilibrium theory can be entirely formulated on the *closed real-time Schwinger-Keldysh contour* (see Fig. 1) with the time argument running from t_0 to ∞ along *time-ordered* branch and back to t_0 along *anti-time-ordered* branch [20]. We assume the reader to be familiar with this real time formulation of the non-equilibrium theory. Details can be found in Refs. [106–108, 111, 112] and in Appendix A.

In absence of pairing one deals only with the “normal” contour Green’s function defined as the expectation value of contour-ordered products of operators

$$i[\hat{G}_n(x, y)]_{ab} = b \xrightarrow{\quad} a = \langle N | \mathcal{T}_C \hat{\Psi}_a(x) \hat{\Psi}_b^\dagger(y) | N \rangle. \quad (1)$$

The time-ordering, \mathcal{T}_C , goes here according to the time parameter running along the time contour \mathcal{C} . The averaging $\langle N | \dots | N \rangle$ is taken over the N -particle non-equilibrium state; a, b are spin indices. In the absence of the spin-orbital interaction the Green’s function is diagonal in the spin space

$$[\hat{G}_n(x, y)]_{ab} = G_n(x, y) \delta_{ab},$$

where δ_{ab} is the Kronecker symbol.

Because of the two contour branches one actually deals with four Green’s functions unified in matrices (so called Schwinger-Keldysh space). Further explanations are given in Appendix A. There one can also find the helpful relations between Green’s functions and self-energies.

The typical interaction time τ_{int} for the change of the higher-order correlation functions is usually much less than the typical relaxation time τ_{rel} , which determines the kinetic stage of the system evolution. Describing the system at times $t - t_0 \gg \tau_{\text{int}}$, one can neglect initial correlations. This is in accordance with the Bogolyubov-Klimontovich principle of the weakening of higher order correlations [109, 110]. The coarse-graining leads to time-irreversibility. Alternatively, one also could suppose that the initial state is uncorrelated, like an in equilibrium ideal gas. This corresponds to a situation when

an information loss occurs right from the beginning, cf. Refs. [19, 111]. Assuming that we describe the system for $t \gg \tau_{\text{int}}$ we may use the Wick decomposition that leads to the Dyson equation formulated on the real time contour

$$\hat{G}_{0x}^{-1} \hat{G}_n(x, y) = \delta_C(x, y) + \int_C dz \hat{\Sigma}_n(x, z) \hat{G}_n(z, y), \quad (2)$$

where

$$\hat{G}_{0x}^{-1} = \left(i \partial_t + \frac{1}{2m} \partial_x^2 \right) \delta_b^a \quad (3)$$

in non-relativistic kinematics that we use in this work. Here δ_C is δ -function on the contour, \hat{G}_0 is the free Green’s function (thin line)

$$\hat{G}_{0x}^{-1} \hat{G}^0(x, y) = \delta_C(x, y), \quad (4)$$

and the self-energy $\hat{\Sigma}_n$ is a functional of the Green’s functions. Being formulated with the standard diagrammatic rules the Dyson equation reads

$$\xrightarrow{\quad} = \xrightarrow{\quad} + \xrightarrow{\quad} \boxed{\Sigma_n} \xrightarrow{\quad}, \quad (5)$$

$$\hat{G}_n = \hat{G}_0 + \hat{G}_0 \odot \hat{\Sigma}_n \odot \hat{G}_n. \quad (6)$$

The sign \odot stands here for the contour coordinate folding, i.e. the integration, $\int_C dz$, over the spatial coordinate and the time coordinate integrated along the Schwinger-Keldysh time contour, cf. Eq. (A2). Alternatively it can be represented as the usual four-dimensional coordinate integration, if all quantities are treated as matrices in the Schwinger-Keldysh space, see Eq. (A7) in Appendix A. Thus, Eq. (6) is the symbolic equation for four (G^{ij} , $i, j = -, +$) Green’s functions. The spin-index contractions go in the standard manner: in the direction opposite to the fermion arrows. Throughout this paper we shall use notations of Refs. [47, 112], in which Σ_n^{+-} and Σ_n^{++} differ by the sign from the corresponding quantities introduced in Ref. [107]. In these notations Eq. (6) is rewritten as

$$\hat{G}_n^{ij} = \hat{G}_0^{ij} + \hat{G}_0^{ik} \odot \hat{\Sigma}_n^{kl} \odot \hat{G}_n^{lj}. \quad (7)$$

The equation for the retarded Green’s function decouples from other equations

$$\hat{G}_n^R = \hat{G}_0^R + \hat{G}_0^R \odot \hat{\Sigma}_n^R \odot \hat{G}_n^R, \quad (8)$$

and reads in diagrams as Eq. (5) (above, this equation was formulated for the contour or matrix quantities). Here $\hat{\Sigma}^R = \hat{\Sigma}^{--} + \hat{\Sigma}^{-+}$ is the retarded self-energy, see (A6). Similar equation exists for the advanced Green function \hat{G}_n^A .

We also need to define two-particle Green’s function

$$\begin{aligned} \hat{K}_n(x, y; x', y') \\ = \langle N | \mathcal{T}_C \hat{\Psi}^c(x') \hat{\Psi}^d(y') \hat{\Psi}_a^\dagger(x) \hat{\Psi}_b^\dagger(y) | N \rangle \end{aligned}$$

and the two-particle interaction amplitude as a contracted part of \hat{K} . The system of equations for the non-equilibrium two-particle Green's functions was studied, e.g., in Refs. [113, 114].

As has been shown by Kadanoff and Baym, in case of smooth time-space changes of the system the quasiclassical approximation can be applied to the non-equilibrium Dyson equations. Using the first-order gradient expansion for the quantities in the Wigner representation one obtains Kadanoff-Baym kinetic equation for the G^{-+} Green's function, which generalizes the standard Boltzmann kinetic equation for quasi-free particle and the Landau kinetic equation for quasi-particle to the case of particles with finite mass-widths. This generalized kinetic equation is supplemented by the equation for the retarded Green's function which is algebraic equation up to second gradients. Self-consistent approximations [115] to this kinetic scheme were developed only recently, see [107, 108, 116] and references therein.

B. NN interaction and pion degrees of freedom in nucleon matter

Consider the nucleon-nucleon interaction formulated within the Fermi-liquid approach with the explicit incorporation of the in-medium pion exchange. For zero temperature it was done by A.B. Migdal in Ref. [27], and, then, the approach was generalized for finite temperatures and non-equilibrium systems in Refs. [28, 32–34, 36, 38]. At excitation energies of our interest ($\epsilon^* \ll \epsilon_F$, $\epsilon^* \sim T$ in equilibrium) nucleons are only slightly excited above their Fermi surfaces and all processes occur in a narrow vicinity of the Fermi energy ϵ_F . Within this approach the long-range processes are treated explicitly, whereas short-range processes are described by local quantities approximated by phenomenological, so-called Landau-Migdal, parameters. At low excitation energies the NN interaction amplitude is presented on the Schwinger-Keldysh contour (or in matrix notation) as follows

$$\text{Diagram (9)} \quad (9)$$

where

$$\text{Diagram (10)} \quad (10)$$

The solid line stands for a nucleon, the double-line stands for a Δ isobar. Although the mass difference between the Δ and N , $m_\Delta - m_N \simeq 2.1m_\pi \gg \epsilon_F$ (m_π is the pion mass) the delta-nucleon hole term is numerically rather large, since the $\pi N\Delta$ coupling constant is twice larger than the πNN one, and the Δ spin-isospin degeneracy factor is 4 times larger than that for nucleons. The doubly-dashed line corresponds to the exchange of the free pion with inclusion of the contributions of the residual s-wave πNN

interaction and $\pi\pi$ scattering, i.e. the residual irreducible interaction to the nucleon particle-hole and delta-nucleon hole insertions. The block in Eq. (10), depicted by the empty square, is irreducible with respect to particle-hole, delta-nucleon hole and pion states and is, by construction, essentially more local than the contributions given by explicitly presented graphs. Thus the empty block term should be much smoother function of its variables than the terms (particle-hole, delta-hole, pion) presented explicitly in Eqs. (9) and (10). In principle, the short-range interaction should be calculated as function of the density, neutron and proton concentrations, energy and momentum, temperature, etc. However, instead of doing complicated calculations one often reduces it to the set of Landau-Migdal parameters, which one extracts from analysis of experimental data on atomic nuclei.

The irreducible part of the interaction involving Δ isobar is constructed similarly to (10)

$$\text{Diagram (11)} \quad (11)$$

The main part of the $N\Delta$ interaction is due to the pion exchange. Although information on local part of the $N\Delta$ interaction is rather scarce, one can conclude [28, 117] that the corresponding Landau-Migdal parameters are essentially smaller than those for NN interaction. Therefore, and also for the sake of simplicity we will, in further, neglect the first graph on the right-hand side of Eq. (11).

The spectrum of the particle excitations is determined by the spectral function given by the imaginary part of the retarded Green's function ($A = -2\Im G^R$). Resummation of diagrams shown in (9) yields the following Dyson equation for the retarded pion Green's function

$$\text{Diagram (12)} \quad (12)$$

Here Σ_{res}^R is the residual retarded pion self-energy that includes the contribution of all diagrams, which are not presented explicitly in (12), like s-wave πN and $\pi\pi$ scatterings (included by doubly-dashed line in (10)). The full vertex takes into account NN correlations

$$\text{Diagram (13)} \quad (13)$$

In some regions of the (ω, k) plane the pion spectral function $A_\pi(\omega, k)$ has sharp peaks along some lines $\omega_i(k)$ which we call the spectral branches. Nearby these lines one can use the quasi-particle approximation² and write

² The quasi-particle approximation for the given particle species is understood as putting the imaginary part of the retarded self-energy in the Green's function to zero. The quasi-particle width $\Gamma = -2\Im\Sigma^R$ is then calculated with so-defined Green's functions.

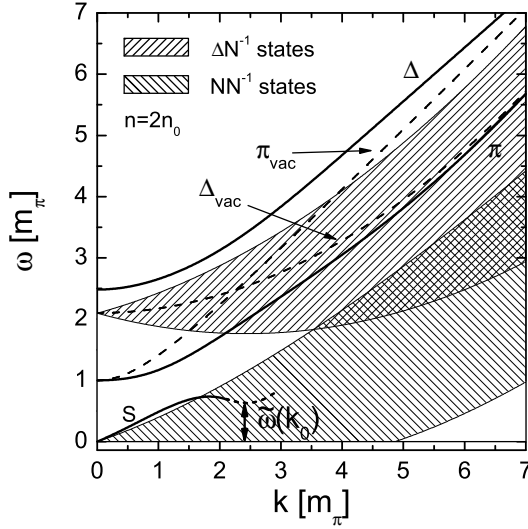


FIG. 2: The typical behavior of the $\pi^{\pm,0}$ spectra in symmetric nuclear matter and of π^0 spectrum in asymmetric matter. The chosen typical value of the density $n = 2n_0$ is supposed to be smaller than the critical density for the pion condensation.

the spectral function as

$$A_\pi(\omega, k) \simeq \sum_i \frac{2\pi \delta(\omega - \omega_i(k))}{\left[2\omega - \frac{\partial \Re \Sigma_\pi^R(\omega, k)}{\partial \omega}\right]_{\omega_i(k)}}. \quad (14)$$

The spectrum of pionic quasi-particles possesses three branches for $\pi^{\pm,0}$ in the symmetric nuclear matter ($N = Z$) and for π^0 in asymmetric matter ($N \neq Z$), e.g. in the neutron star matter. The typical spectrum is shown in Fig. 2. In the region $\omega \gtrsim m_\pi$ there are two branches: the Δ branch and the pion branch. For $\omega < m_\pi$ there is the spin sound branch (with $\omega \rightarrow 0$ when $k \rightarrow 0$). The hatches show the regions on the (ω, k) plane with a non-vanishing pion width, calculated within the quasi-particle approximation for nucleons and Δ isobars. In the lower hatched region, at $\omega < kp_{F,N}$ and $k \sim p_{F,N}$, there are no quasi-particle branches and the pion width cannot be neglected. This is the region of the Landau damping in the nucleon particle-hole channel. The pion spectral function is enhanced in this region of ω and k for $n > n_{c1} \sim (0.5-0.7)n_0$. We stress that the pion spectral function calculated beyond the quasi-particle approximation for nucleons and Δ 's is much more involved, see Ref. [118].

To specify the enhancement of the $\pi^{\pm,0}$ spectral density for $N = Z$ and of the π^0 for $N \neq Z$ in the Landau damping region it is convenient to introduce the function

$$\tilde{\omega}^2(k) = -[\Re G_\pi^R(\omega = 0, k, \mu_\pi)]^{-1}.$$

Note that momenta passing through the NN interaction in the MU and MMU processes are $k \sim p_{F,n}$, where $p_{F,n}$ is the nucleon Fermi momentum, and for the MNB process $k \sim (0.9-1)p_{F,n}$ [46]. Remarkably, the minimum on

the function $\tilde{\omega}^2(k)$ is realized at the similar momentum $k_0 \simeq (0.9-1)p_{F,n}$. Thus, the quantity $\tilde{\omega}(k_0)$ called the *effective pion gap* controls the strength of the NN interaction. The NN cross-section is $\propto 1/\tilde{\omega}^4$ for $\tilde{\omega}^2 < m_\pi^2$, provided the MOPE dominates, see Eq. (15) below. Note that for the asymmetric nucleon matter the pion gap is different for π^0 and for π^\pm since neutral and charged channels are characterized by different diagrams permitted by the charge conservation.

The pion chemical potentials ($\mu_{\pi^+} \neq \mu_{\pi^-} \neq 0$, $\mu_{\pi^0} = 0$) are determined from equilibrium conditions for the reactions involving the corresponding pions. In the neutron star matter μ_{π^-} follows from the condition of the chemical quasi-equilibrium with respect to the reactions $n \rightarrow p\pi^-$ and $n \rightarrow p\bar{\nu}$: $\mu_{\pi^-} = \mu_e = \varepsilon_{F,n} - \varepsilon_{F,p}$, where $\varepsilon_{F,n}$, $\varepsilon_{F,p}$ are Fermi energies of the neutron and proton. For a small-size systems like atomic nucleus one should put $\mu_{\pi^+} = \mu_{\pi^-} = \mu_{\pi^0} = 0$.

At low pion energies (for $\pi^{\pm,0}$ for $N = Z$ and for π^0 for $N \neq Z$) the lowest-energy state determining by the pole of the pion Green's function is $i\beta\omega \simeq \tilde{\omega}^2(k_0)$ with $\beta > 0$ appeared due to the Landau damping, see Ref. [28]. Thus for $\tilde{\omega}^2(k_0) > 0$ the pion excitations die out with time exponentially $\propto \exp(-\tilde{\omega}^2(k_0)t/\beta)$.

C. Pion softening and pion condensation

For $\tilde{\omega}^2(k_0) < 0$ the pion field grows exponentially with time as $\exp(|\tilde{\omega}^2(k_0)|t/\beta)$. Thus, the change in the sign of $\tilde{\omega}^2$ at $n = n_{c,\pi}$ marks the critical point of a phase transition to a state with a classical pion field (a pion condensate). The critical density $n_{c,\pi}$ depends on the values of Landau-Migdal parameters, which are badly known for asymmetric matter and for densities significantly larger than n_0 . Nevertheless, some estimations can be given. Various experiments have shown that the pion condensation does not manifest itself in atomic nuclei as a volume effect, see Refs. [27, 28]³. Different model-dependent estimations indicate that $n_{c,\pi} \sim (1.5-3)n_0$, depending on the pion species, the proton-to-neutron ratio and the model used, see Refs. [28, 119]. Variational calculations [121] yield $n_{c,\pi} \simeq 2n_0$ for isotopically symmetric nuclear matter and $n_{c,\pi} \simeq 1.3n_0$ for π^0 mesons in the neutron star matter.

Typical density behavior of $\tilde{\omega}^2(k_0)$ (for π^\pm, π^0 at $N = Z$ and for π^0 at $N \gg Z$) is shown in Fig. 3. At $n < n_{c1}$, $\tilde{\omega}^2(k)$ has the minimum for $k_0 = 0$, i.e. $\tilde{\omega}^2(k_0 = 0) = m_\pi^2 - \mu_\pi^2$. For such densities the value $\tilde{\omega}^2(p_{F,n})$ essentially deviates from $m_\pi^2 - \mu_\pi^2$ tending to $m_\pi^2 + p_{F,n}^2 - \mu_\pi^2$ in the low density limit.

At the critical point of the pion condensation ($n =$

³ In the surface layer of the nucleus a classical pion condensate field may appear due to a coupling of the gradient of the σ mean field to the pion, see [120].

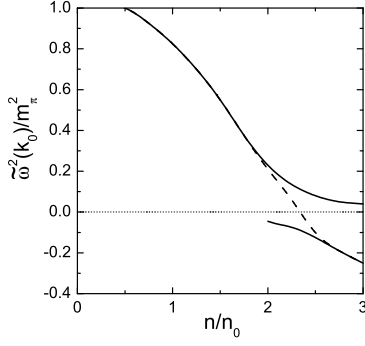


FIG. 3: Effective pion gap (for $\pi^{\pm,0}$ for $N = Z$ and for π^0 for $N \neq Z$) versus the baryon density as calculated in Refs. [28, 38]. Compare with the pion spectrum presented in Fig. 2.

$n_{c,\pi}$) the value $\tilde{\omega}^2(k_0)$ with artificially neglected $\pi\pi$ fluctuations changes its sign (dashed line in Fig. 3). It symbolizes the occurrence of a second-order phase transition to an inhomogeneous ($k_0 \neq 0$) pion-condensate state. In reality, the $\pi\pi$ fluctuations are significant in the vicinity of the critical point and the phase transition is of the first order [33, 34, 122]. Therefore we depict two branches in Fig. 3 (solid curves) with positive and negative values of $\tilde{\omega}^2(k_0)$. Calculations in Ref. [34] demonstrated that at $n > n_{c,\pi}$ the free energy of the state with $\tilde{\omega}^2(k_0) > 0$ and without the pion mean field becomes larger than the free energy of the state with $\tilde{\omega}^2(k_0) < 0$ and a finite pion mean field. Therefore at $n > n_{c,\pi}$ the state with $\tilde{\omega}^2(k_0) > 0$ is metastable and the state with $\tilde{\omega}^2(k_0) < 0$ and the pion mean field $\varphi_\pi \neq 0$ becomes the ground state.

The quantity $\tilde{\omega}^2(k_0)$ demonstrates how much the virtual (particle-hole) mode with pion quantum numbers is softened at the given density. For the symmetric nuclear matter at $n = n_0$ the ratio $\alpha = G_\pi/G_\pi^0 \simeq 6$ for $\omega = 0$, $k = p_{F,N}$. However, this so-called “pion softening” [27] does not significantly enhance the NN scattering cross section because of the simultaneous essential suppression of the πNN vertex by nucleon-nucleon correlations. Indeed, the ratio of the NN cross sections calculated with the FOPE and MOPE equals to

$$R = \frac{\sigma[\text{FOPE}]}{\sigma[\text{MOPE}]} \simeq \frac{\gamma^4(\omega \simeq 0, k \simeq p_{F,N})(m_\pi^2 + p_{F,N}^2)^2}{\tilde{\omega}^4(p_{F,N})}, \quad (15)$$

where γ is the vertex dressing factor determined by Eq. (13), $\gamma(n_0) \simeq 0.4$. For $n \lesssim n_0$ one has $R \lesssim 1$, whereas already for $n = 2n_0$ this estimate yields $R \sim 10$. Thus, following Refs. [27, 28] one can evaluate the NN interaction for $n > n_0$ with the help of the MOPE, i.e.,

$$(16)$$

Here the bold wavy line relates to the in-medium pion. In the soft-pion approximation the same one-pion exchange determines also interaction in the particle-particle chan-

nel

$$(17)$$

Namely, this quantity determines the NN interaction entering neutrino emissivities of the MMU and MNB processes.

Often, one considers the softening of the one-pion exchange only, neglecting the same effects for the two, three etc. pion exchanges, arguing for their smallness because of extra integrations over the intermediate states, see Ref. [27]. At zero temperature these effects are numerically small. Nevertheless, they result in the change of the order of the phase transition (from the second order to the first order) [122]. Ignoring this small jump in the pion gap one may deal with the MOPE interaction in the one (particle-hole or particle-particle) channel. The residual interactions are then hidden in the values of Landau-Migdal parameters. In case of the non-equilibrium and equilibrium $T \neq 0$ matter these pion fluctuation effects contribute significantly to the pion self-energy and should be taken into account, see Refs. [33, 34]. Corresponding pion fluctuation contributions must be then extracted from the residual interaction parameters. Their self-consistent analysis in both particle-hole and particle-particle channels has been performed in Ref. [34] within the Thomas-Fermi approximation ($k_0 \ll 2p_{F,N}$).

III. DESCRIPTION OF NON-EQUILIBRIUM SUPERFLUID FERMION LIQUIDS

Below, dealing with pairing phenomena in non-equilibrium systems we assume that deviations from the equilibrium are rather small, bearing in mind that non-equilibrium effects should not destroy the fermion pairing.

In a superfluid Fermi system a condensate of paired fermions is formed. It induces non-vanishing amplitudes for the transitions of a particle to a hole state and vice versa. Thus, it is possible to combine one-particle state on top of the N -particle background and one-hole state on top of the background with $N+2$ particles. The one-particle-one-hole irreducible amplitudes of such transitions are depicted as blocks

$$\begin{aligned} -i[\hat{\Delta}^{(1)}(x)]_{ab} &= -i \langle N | \hat{\Psi}_a(x) \hat{\Psi}_b(x) | N+2 \rangle \\ &= b \leftarrow \hat{\Delta}^{(1)} \rightarrow a, \\ -i[\hat{\Delta}^{(2)}(x)]_{ab} &= -i \langle N+2 | \hat{\Psi}_a^\dagger(x) \hat{\Psi}_b^\dagger(x) | N \rangle \\ &= b \rightarrow \hat{\Delta}^{(2)} \leftarrow a. \end{aligned} \quad (18)$$

Their spin structure can be written in general case as

$$\begin{aligned} \hat{\Delta}^{(1)}(x) &= (\Delta_0^{(1)}(x) \sigma_0 + \vec{\Delta}_1^{(1)}(x) \vec{\sigma}) i \sigma_2, \\ \hat{\Delta}^{(2)}(x) &= i \sigma_2 (\Delta_0^{(2)}(x) \sigma_0 + \vec{\Delta}_1^{(2)}(x) \vec{\sigma}). \end{aligned} \quad (19)$$

Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with σ_i being the Pauli matrices; σ_0 is the unit matrix in the spin space. The amplitudes $\Delta_0^{(1,2)}$ are induced by a particle-particle interaction with even angular momenta $L = 0, 2, 4, \dots$, and amplitudes $\bar{\Delta}_1^{(1,2)}$, by the interaction with odd angular momenta $L = 1, 3, 5, \dots$.

Since $\hat{\Delta}^{(1,2)}$ are functions of the only one contour coordinate, we have $[\hat{\Delta}^{(1,2)}]^{-+} = [\hat{\Delta}^{(1,2)}]^{+-} = 0$ and denote $[\Delta^{(1,2)}]^{--} = [\Delta^{(1,2)}]^R = -[\Delta^{(1,2)}]^{++}$, as follows from Eq. (A5) valid for any two-point functions. From the definitions (18) follows

$$(\hat{\Delta}^{(1)}(x))^\dagger = (\hat{\Delta}^{(2)}(x))^T, \quad (20)$$

and Eq. (19) yields

$$(\Delta_0^{(1)})^* = \Delta_0^{(2)}, \quad (\bar{\Delta}_1^{(1)})^* = -\bar{\Delta}_1^{(2)}. \quad (21)$$

The transition amplitudes (18) imply that a propagating particle can be transformed in flight into a hole and vice versa. This is described by a new type of propagators — anomalous Green's functions — which can be defined on the Schwinger-Keldysh contour as

$$\begin{aligned} [i\hat{F}^{(1)}(x, y)]_{ab} &= b \overleftrightarrow{\hspace{0.5cm}} a \\ &= \langle N | \mathcal{T}_C \hat{\Psi}_a(x) \hat{\Psi}_b(y) | N+2 \rangle, \\ [i\hat{F}^{(2)}(x, y)]_{ab} &= b \overleftarrow{\hspace{0.5cm}} a \\ &= \langle N+2 | \mathcal{T}_C \hat{\Psi}_a^\dagger(x) \hat{\Psi}_b^\dagger(y) | N \rangle. \end{aligned} \quad (22)$$

For these two functions are valid relations

$$[\hat{F}^{(1,2)}(x, y)]_{ab} = -[\hat{F}^{(1,2)}(y, x)]_{ba}, \quad (23)$$

$$[\hat{F}^{(1,2)}(x, y)]_{ab}^\dagger = -[\hat{F}^{(2,1)}(y, x)]_{ba}. \quad (24)$$

We also need the Green's function for a hole

$$\begin{aligned} [i\hat{G}^h(x, y)]_{ab} &= a \overleftarrow{\hspace{0.5cm}} b \\ &= \langle N | \mathcal{T}_C \hat{\Psi}_{C,a}(x) \hat{\Psi}_{C,b}^\dagger(y) | N \rangle, \end{aligned} \quad (25)$$

which is written in terms of the charge conjugated fermion operators $\hat{\Psi}_C = \sigma_2 (\hat{\Psi}^\dagger)^T$, where T stands for the transposition operation in the spin space. The Green's functions of a particle and a hole are related as

$$\hat{G}^h(x, y) = \sigma_2 [\hat{G}(y, x)]^T \sigma_2 = G(y, x) \delta_{ab}. \quad (26)$$

The Dyson equations for dressed normal and anomalous Green's functions in case of pairing (Gor'kov equations) have the form:

$$\overrightarrow{\hspace{0.5cm}} = \overrightarrow{\hspace{0.5cm}} + \overrightarrow{\hspace{0.5cm}} \boxed{\Sigma} \overrightarrow{\hspace{0.5cm}} + \overrightarrow{\hspace{0.5cm}} \triangle^{\Delta^{(1)}} \overrightarrow{\hspace{0.5cm}}, \quad (27)$$

$$\overleftarrow{\hspace{0.5cm}} = \overleftarrow{\hspace{0.5cm}} + \overleftarrow{\hspace{0.5cm}} \boxed{\Sigma} \overleftarrow{\hspace{0.5cm}} + \overleftarrow{\hspace{0.5cm}} \triangle^{\Delta^{(2)}} \overleftarrow{\hspace{0.5cm}}, \quad (28)$$

$$\overleftrightarrow{\hspace{0.5cm}} = \overleftrightarrow{\hspace{0.5cm}} + \overleftrightarrow{\hspace{0.5cm}} \boxed{\Sigma} \overleftrightarrow{\hspace{0.5cm}} + \overleftrightarrow{\hspace{0.5cm}} \triangle^{\Delta^{(1)}} \overleftrightarrow{\hspace{0.5cm}}. \quad (29)$$

The thin line is the normal free Green's function of the fermion particle and the inverted thin line is the normal free Green's function of the hole, $\hat{\Sigma}$ is the full self-energy of the particle and $\hat{\Sigma}^h$, of the hole. Explicitly these Dyson equations read as

$$\begin{aligned} \hat{G} &= \hat{G}_0 + \hat{G}_0 \odot \hat{\Sigma} \odot \hat{G} + \hat{G}_0 \odot \hat{\Delta}^{(1)} \odot \hat{F}^{(2)}, \\ \hat{F}^{(2)} &= \hat{G}_0^h \odot \hat{\Sigma}^h \odot \hat{F}^{(2)} + \hat{G}_0^h \odot \hat{\Delta}^{(2)} \odot \hat{G}, \\ \hat{F}^{(1)} &= \hat{G}_0 \odot \hat{\Sigma} \odot \hat{F}^{(1)} + \hat{G}_0 \odot \hat{\Delta}^{(1)} \odot \hat{G}^h. \end{aligned} \quad (30)$$

In terms of the dressed Green's functions for the system without pairing (6) the Gor'kov equations can be written shortly as [15],

$$\begin{aligned} \hat{G} &\simeq \hat{G}_n + \hat{G}_n \odot \hat{\Delta}^{(1)} \odot \hat{F}^{(2)}, \\ \hat{F}^{(2)} &\simeq \hat{G}_n^h \odot \hat{\Delta}^{(2)} \odot \hat{G}, \\ \hat{F}^{(1)} &\simeq \hat{G}_n \odot \hat{\Delta}^{(1)} \odot \hat{G}^h. \end{aligned} \quad (31)$$

In these equations we dropped the terms containing the differences $(\hat{\Sigma} - \hat{\Sigma}_n)$ and $(\hat{\Sigma}^h - \hat{\Sigma}_n^h)$. The self-energy Σ includes the same set of diagrams as the self-energy Σ_n , but constructed now from the Green's functions for the superfluid system, G and F , instead of the normal Green's function G_n . Note that the anomalous Green's functions F can enter Σ only in pairs to preserve the number of incoming and outgoing fermion lines in each vertex. Hence, the terms containing F 's are $\propto \Delta^2$. Since in the momentum representation $G - G_n \propto \Delta^2$ and $G \rightarrow G_n$ for $|p_0 - \epsilon_F|/\Delta \gg 1$, each integral over the energy p_0 in the self-energy is accumulated only in the vicinity of the Fermi surface. Thus the neglected terms are small as Δ^2/ϵ_F^2 , cf. also Ref. [15].

Hereafter the thin line in diagrams will stand for the full Green's function of the system without pairing. From Eq. (31) it follows immediately that the Green's function G is diagonal in the spin space. Then the Green's functions $F^{(1)}$ and $F^{(2)}$ have the same spin structure as the blocks (18):

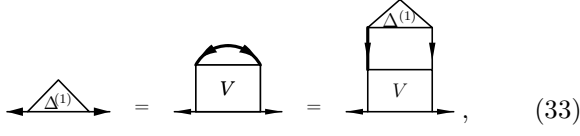
$$\begin{aligned} \hat{F}^{(1)}(x, y) &= (F_0^{(1)}(x, y) \sigma_0 + \bar{F}_1^{(1)}(x, y) \vec{\sigma}) i\sigma_2, \\ \hat{F}^{(2)}(x, y) &= i\sigma_2 (F_0^{(2)}(x, y) \sigma_0 + \bar{F}_1^{(2)}(x, y) \vec{\sigma}). \end{aligned} \quad (32)$$

In further, we will assume that only one type of pairing may occur: either in a state with an even angular momentum or in a state with an odd angular momentum.⁴ Then we have either only $\Delta_0^{(1,2)} \neq 0$ or only $\bar{\Delta}_1^{(1,2)} \neq 0$, and correspondingly, either $F_0^{(1,2)} \neq 0$ or $\bar{F}_1^{(1,2)} \neq 0$, but not both simultaneously. In this case the Green's function \hat{G} remains diagonal in the spin space. Then we may follow derivations of [15] with a little difference that

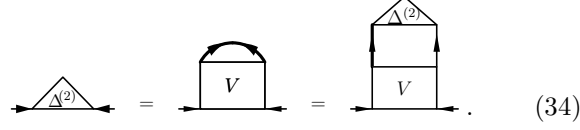
⁴ It is however not excluded that at some conditions a part of fermions can be paired in one state whereas other part, in another state.

all our expressions are valid on the Schwinger-Keldysh contour.

The graphical equations for $\Delta^{(1)}$ and $\Delta^{(2)}$ are

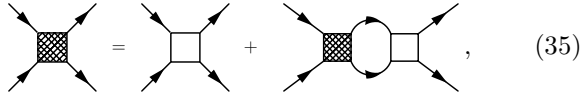


$$\Delta^{(1)} = V = \hat{V}, \quad (33)$$



$$\Delta^{(2)} = V = \hat{V}. \quad (34)$$

Here \hat{V} is a two-particle irreducible interaction, that determines the full in-medium particle-particle (“pp”) scattering amplitude via graphical equation



$$\hat{T}_{pp} = \hat{V} - i \hat{V} \square (\hat{G}_n \hat{G}_n) \square \hat{T}_{pp}, \quad (35)$$

which in terms of contour foldings becomes

$$\hat{T}_{pp} = \hat{V} - i \hat{V} \square (\hat{G}_n \hat{G}_n) \square \hat{T}_{pp}. \quad (36)$$

Both \hat{V} and \hat{T} are to be understood as formulated on the Schwinger-Keldysh contour,

$$\begin{aligned} \hat{T}_{pp}(x', y'; x, y) &= \hat{V}(x', y'; x, y) \\ &- i \int_C dz_1 \int_C dz_2 \int_C dz_3 \int_C dz_4 \hat{V}(x', y'; z_1, z_2) \\ &\times \hat{G}_n(z_1, z_3) \hat{G}_n(z_2, z_4) \hat{T}_{pp}(z_3, z_4; x, y), \end{aligned} \quad (37)$$

or being matrices in the Schwinger-Keldysh space. The sign \square in (36) stands for integration over two four-dimensional coordinates with the time running over the Schwinger-Keldysh contour. The quantities \hat{V} and \hat{T} are also matrices in the spin space, and we parameterize \hat{V} as

$$\begin{aligned} \hat{V}_{cd,ab} &= i \begin{array}{c} b \quad d \\ \hline V \\ \hline a \quad c \end{array} \\ &= V_0(i\sigma_2)_{dc}(i\sigma_2)_{ab} + V_1^{\alpha\beta}(\vec{\sigma}^\alpha i\sigma_2)_{dc}(i\sigma_2 \vec{\sigma}^\beta)_{ab}. \end{aligned} \quad (38)$$

With this definition the interactions V_0 and V_1 correspond to the scattering of two fermions with the total spin zero and one, respectively. The interaction block entering Eq. (33) is given by

$$\begin{array}{c} b \quad d \\ \hline V \\ \hline a \quad c \end{array} = \hat{V}_{ab,cd}. \quad (39)$$

Using this definition we can analyze the spin structure of Eq. (34)

$$\begin{aligned} [\hat{\Delta}^{(2)}]_{ab} &= -i[\hat{F}^{(2)}]_{cd} \square \hat{V}_{cd,ab} \\ &= -i[\hat{G}_n^h]_{c\bar{c}} \odot [\hat{\Delta}^{(2)}]_{cd} \odot \hat{G}_{dd} \square \hat{V}_{cd,ab}. \end{aligned} \quad (40)$$

Substituting here Eqs. (19,32) and (38) and taking into account that the full Green’s functions G and G_n are diagonal in the spin-space, we obtain

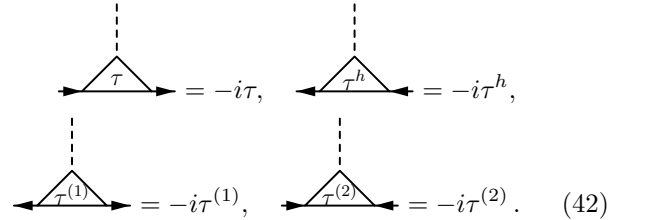
$$\begin{aligned} &[i\sigma_2(\Delta_0^{(2)}\sigma_0 + \vec{\Delta}_1^{(2)}\vec{\sigma})]_{ab} \\ &= -i[G_n^h \odot i\sigma_2(\Delta_0^{(2)}\sigma_0 + \vec{\Delta}_1^{(2)}\vec{\sigma}) \odot G]_{cd} \\ &\square[V_0(i\sigma_2)_{dc}(i\sigma_2)_{ab} + V_1^{\alpha\beta}(\vec{\sigma}^\alpha i\sigma_2)_{dc}(i\sigma_2 \vec{\sigma}^\beta)_{ab}]. \end{aligned}$$

Separating terms with σ_0 and $\vec{\sigma}$ Pauli matrices and making use of the relations $(i\sigma_2)^2 = -1$ and $\text{Tr}\{\sigma^\alpha \sigma^\beta\} = 2\delta^{\alpha\beta}$ we find

$$\begin{aligned} \Delta_0^{(2)} &= 2i(G_n^h \odot \Delta_0^{(2)} \odot G) \square V_0, \\ \vec{\Delta}_1^{(2)\alpha} &= 2i(G_n^h \odot \Delta_1^{(2)\beta} \odot G) \square V_1^{\beta\alpha}. \end{aligned} \quad (41)$$

For $\Delta_0^{(1)}$ and $\vec{\Delta}_1^{(1)}$ we obtain exactly the same equations as (41) but with the replacement $G_n^h \leftrightarrow G$.

An external field V^{ext} acting on a superfluid Fermi system can induce four different effects determined by the four vertex functions related to the creation of particle and hole τ , anti-particle and anti-hole τ^h , two particles $\tau^{(2)}$, and two holes $\tau^{(1)}$. The vertices can be graphically depicted as



$$\begin{aligned} \tau &= -i\tau, & \tau^h &= -i\tau^h, \\ \tau^{(1)} &= -i\tau^{(1)}, & \tau^{(2)} &= -i\tau^{(2)}. \end{aligned} \quad (42)$$

In matrix notations, vertices have three indices, $\hat{\tau}_k^{ij}$, where the lower index relates to the external dash line. The couplings of the external field to the particle and to the hole are related as

$$\hat{\tau}^h(x, z, y) = [\hat{\tau}(y, z, x)]^T. \quad (43)$$

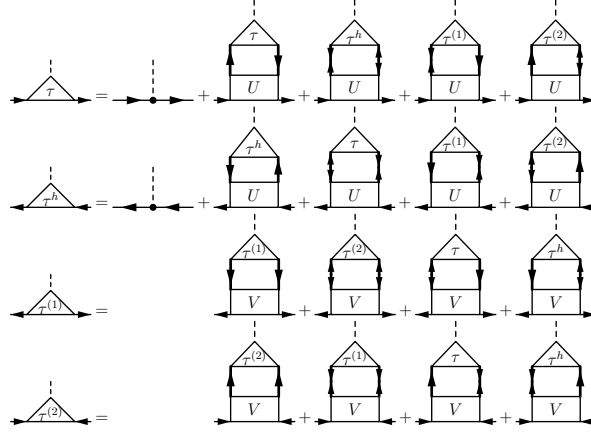
The coupling of an external field to the non-relativistic fermion is described by the 2×2 matrix acting in the fermion spin space. Any rank-2 matrix can be decomposed into a unit matrix σ_0 and Pauli matrices $\vec{\sigma}$. Thus, we have

$$\begin{aligned}
\hat{\tau} &= t_0 \sigma_0 + \vec{\sigma} \vec{t}_1, \quad \hat{\tau}^h = t_0^h \sigma_0 + \vec{\sigma}^T \vec{t}_1^h, \\
\hat{\tau}^{(1)} &= (t_0^{(1)} \sigma_0 + \vec{\sigma} \vec{t}_1^{(1)}) i \sigma_2, \\
\hat{\tau}^{(2)} &= i \sigma_2 (t_0^{(2)} \sigma_0 + \vec{\sigma} \vec{t}_1^{(2)}),
\end{aligned} \tag{44}$$

and similarly for the bare vertices

$$\hat{\tau}_0 = t_{0,0} \sigma_0 + \vec{\sigma} \vec{t}_{0,1}, \quad \hat{\tau}_0^h = t_{0,0}^h \sigma_0 + \vec{\sigma}^T \vec{t}_{0,1}^h. \tag{45}$$

The vertices obey the following equations defined on the Schwinger-Keldysh contour:



$$\begin{aligned}
\triangle_{\tau} &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \\
\triangle_{\tau^h} &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \\
\triangle_{\tau^{(1)}} &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \\
\triangle_{\tau^{(2)}} &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---}
\end{aligned} \tag{46}$$

Intermediate lines in (46) are of all possible signs in the Schwinger-Keldysh space. In operator form equations for vertices read

$$\hat{\tau} = \hat{\tau}_0 - i \left[\hat{G} \odot \hat{\tau} \odot \hat{G} + \hat{F}^{(1)} \odot \hat{\tau}^h \odot \hat{F}^{(2)} + \hat{G} \odot \hat{\tau}^{(1)} \odot \hat{F}^{(2)} + \hat{F}^{(1)} \odot \hat{\tau}^{(2)} \odot \hat{G} \right] \square \hat{U}, \tag{47a}$$

$$\hat{\tau}^h = \hat{\tau}_0^h - i \left[\hat{G}^h \odot \hat{\tau}^h \odot \hat{G}^h + \hat{F}^{(2)} \odot \hat{\tau} \odot \hat{F}^{(1)} + \hat{F}^{(2)} \odot \hat{\tau}^{(1)} \odot \hat{G}^h + \hat{G}^h \odot \hat{\tau}^{(2)} \odot \hat{F}^{(1)} \right] \square \hat{U}, \tag{47b}$$

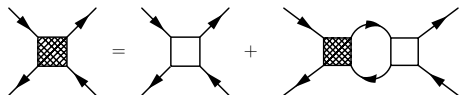
$$\hat{\tau}^{(1)} = -i \left[\hat{G} \odot \hat{\tau}^{(1)} \odot \hat{G}^h + \hat{F}^{(1)} \odot \hat{\tau}^{(2)} \odot \hat{F}^{(1)} + \hat{G} \odot \hat{\tau} \odot \hat{F}^{(1)} + \hat{F}^{(1)} \odot \hat{\tau}^h \odot \hat{G}^h \right] \square \hat{V}, \tag{47c}$$

$$\hat{\tau}^{(2)} = -i \left[\hat{G}^h \odot \hat{\tau}^{(2)} \odot \hat{G} + \hat{F}^{(2)} \odot \hat{\tau}^{(1)} \odot \hat{F}^{(2)} + \hat{F}^{(2)} \odot \hat{\tau} \odot \hat{G} + \hat{G}^h \odot \hat{\tau}^h \odot \hat{F}^{(2)} \right] \square \hat{V}. \tag{47d}$$

Here U is the particle-hole irreducible interaction, which determines the full particle-hole scattering amplitude via the equation

$$\hat{T}_{\text{ph}} = \hat{U} - i \hat{U} \square (\hat{G} \hat{G}^h) \square \hat{T}_{\text{ph}}. \tag{48}$$

In diagrams this equation has the form



$$\text{---} \text{---} \text{---} \text{---} = \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---}, \tag{49}$$

which differs from (35) by inversion of one of the nucleon lines. In the spin space the matrix \hat{U} is defined as

$$\begin{aligned}
\hat{U}_{dc,ab} &= i \text{---} \text{---} \text{---} \text{---} \\
&= U_0 \delta_{dc} \delta_{ab} + U_1^{\alpha\beta} \vec{\sigma}_{dc}^{\beta} \vec{\sigma}_{ab}^{\alpha}.
\end{aligned} \tag{50}$$

The amplitudes U_0 and U_1 are determined on the contour or they are matrices in the Schwinger-Keldysh space.

Having at our disposal the complete spin structure of all elements we can work out the spin algebra in Eqs. (47). As an illustration we consider the second and third terms in Eq. (47a):

$$[(t_0 - t_{0,0}) \sigma_0 + \vec{\sigma} (\vec{t}_1 - \vec{t}_{0,1})]_{ab} = -i[(F_0^{(1)} \sigma_0 + \vec{F}_1^{(1)} \vec{\sigma}) i\sigma_2 \odot (t_0^h \sigma_0 + \vec{\sigma}^T \vec{t}_1^h) \odot i\sigma_2 (F_0^{(2)} \sigma_0 + \vec{F}_1^{(2)} \vec{\sigma}) \\ + G\sigma_0 \odot (t_0^{(1)} \sigma_0 + \vec{\sigma} \vec{t}_1^{(1)}) i\sigma_2 \odot i\sigma_2 (F_0^{(2)} \sigma_0 + \vec{F}_1^{(2)} \vec{\sigma}) + \dots]_{cd} \square (U_0 \delta_{dc} \delta_{ab} + U_1^{\alpha\beta} \vec{\sigma}_{dc}^\beta \vec{\sigma}_{ab}^\alpha).$$

For the spin-scalar t_0 component we obtain

$$t_0 - t_{0,0} = -2i \left[- (F_0^{(1)} \odot t_0^h \odot F_0^{(2)} + \vec{F}_1^{(1)} \odot t_0^h \odot \vec{F}_1^{(2)}) - G \odot (t_0^{(1)} \odot F_0^{(2)} + \vec{t}_1^{(1)} \odot \vec{F}_1^{(2)}) + \dots \right] \square U_0. \quad (51)$$

We dropped here all terms containing simultaneously $F_0^{(1,2)}$ and $\vec{F}_1^{(1,2)}$ Green's functions, since we assumed only one type of pairing in our system. Using the trace $\text{Tr}\{\sigma^\alpha \sigma^\beta \sigma^\gamma\} = i 2 \epsilon_{\alpha\beta\gamma}$ we obtain the term $i \epsilon_{\alpha\beta\gamma} \vec{F}_1^{(1)\alpha} \odot \vec{t}_1^{h(1)\beta} \odot \vec{F}_1^{(2)\gamma}$, which vanishes since the vectors $\vec{F}_1^{(1)}$ and $\vec{F}_1^{(2)}$ are colinear and may differ only by a phase, see Eq. (24).

The equation for the spin-vector vertex \vec{t}_1 is more involved

$$\vec{t}_1^\alpha - \vec{t}_{0,1}^\alpha = -2i \left[- (F_0^{(1)} \odot \vec{t}_1^{h\beta} \odot F_0^{(2)} - (\delta_{\gamma\delta} \delta_{\lambda\beta} - \delta_{\gamma\lambda} \delta_{\delta\beta} + \delta_{\gamma\beta} \delta_{\delta\lambda}) \vec{F}_1^{(1)\gamma} \odot \vec{t}_1^{h\delta} \odot \vec{F}_1^{(2)\lambda}) \right. \\ \left. - G \odot (\vec{t}_1^{(1)\beta} \odot F_0^{(2)} + (t_0^{(1)} \delta_{\beta\delta} + i \epsilon_{\beta\gamma\delta} \vec{t}_1^{(1)\gamma}) \odot \vec{F}_1^{(2)\delta}) + \dots \right] \square U_1^{\alpha\beta}. \quad (52)$$

We used here the trace $\text{Tr}\{\sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta\} = 2 (\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma})$. The different signs of terms in the first bracket appear because $\sigma_2 \vec{\sigma}^T \sigma_2 = -\vec{\sigma}$. We elaborate the spin structure of V at hand of the second and fourth terms in Eq. (47c):

$$[(t_0^{(1)} \sigma_0 + \vec{\sigma} \vec{t}_1^{(1)}) i\sigma_2]_{ab} = -i[(F_0^{(1)} \sigma_0 + \vec{F}_1^{(1)} \vec{\sigma}) i\sigma_2 \odot i\sigma_2 (t_0^{(2)} \sigma_0 + \vec{\sigma} \vec{t}_1^{(2)}) \odot (F_0^{(1)} \sigma_0 + \vec{F}_1^{(1)} \vec{\sigma}) i\sigma_2 \\ + (F_0^{(1)} \sigma_0 + \vec{F}_1^{(1)} \vec{\sigma}) i\sigma_2 \odot (t_0^h \sigma_0 + \vec{\sigma}^T \vec{t}_1^h) \odot G^h \sigma_0 + \dots]_{cd} (V_0 (i\sigma_2)_{dc} (i\sigma_2)_{ab} + V_1^{\alpha\beta} (i\sigma_2 \vec{\sigma}^\beta)_{dc} (\vec{\sigma}^\alpha i\sigma_2)_{ab}).$$

This equation reduces to

$$t_0^{(1)} = -2i \left[+ (F_0^{(1)} \odot t_0^{(2)} \odot F_0^{(1)} + \vec{F}_1^{(1)} \odot t_0^{(2)} \odot \vec{F}_1^{(1)}) - (F_0^{(1)} \odot t_0^h - \vec{F}_1^{(1)} \odot \vec{t}_1^h) \odot G^h + \dots \right] \square V_0, \quad (53)$$

$$\vec{t}_1^{(1)\alpha} = -2i \left[+ (F_0^{(1)} \odot \vec{t}_1^{(2)\beta} \odot F_0^{(1)} + (\delta_{\gamma\delta} \delta_{\lambda\beta} - \delta_{\gamma\lambda} \delta_{\delta\beta} + \delta_{\gamma\beta} \delta_{\delta\lambda}) \vec{F}_1^{(1)\gamma} \odot \vec{t}_1^{(2)\delta} \odot \vec{F}_1^{(1)\lambda}) \right. \\ \left. - (\vec{F}_1^{(1)\gamma} \odot (t_0^h \delta_{\gamma\beta} - i \epsilon_{\beta\gamma\delta} \vec{t}_1^{h\delta}) - F_0^{(1)} \odot \vec{t}_1^{h\beta}) \odot G^h + \dots \right] \square V_1^{\alpha\beta}. \quad (54)$$

Proceeding this way and collecting all terms we rewrite Eqs. (47) for the scalar vertices as

$$t_0 - t_{0,0} = -2i \left[G \odot \tau_0 \odot G - F_0^{(1)} \odot t_0^h \odot F_0^{(2)} - G \odot t_0^{(1)} \odot F_0^{(2)} - F_0^{(1)} \odot t_0^{(2)} \odot G \right. \\ \left. - \vec{F}_1^{(1)} \odot t_0^h \odot \vec{F}_1^{(2)} - G \odot \vec{t}_1^{(1)} \odot \vec{F}_1^{(2)} - \vec{F}_1^{(2)} \odot \vec{t}_1^{(1)} \odot G \right] \square U_0, \quad (55a)$$

$$t_0^h - t_{0,0}^h = -2i \left[G^h \odot \tau_0^h \odot G^h - F_0^{(2)} \odot t_0 \odot F_0^{(1)} - F_0^{(2)} \odot t_0^{(1)} \odot G^h - G^h \odot t_0^{(2)} \odot F_0^{(1)} \right. \\ \left. - \vec{F}_1^{(2)} \odot t_0 \odot \vec{F}_1^{(1)} - \vec{F}_1^{(2)} \odot \vec{t}_1^{(1)} \odot G^h - G^h \odot \vec{t}_1^{(2)} \odot \vec{F}_1^{(1)} \right] \square U_0, \quad (55b)$$

$$t_0^{(1)} = +2i \left[G \odot t_0^{(1)} \odot G^h - F_0^{(1)} \odot t_0^{(2)} \odot F_0^{(1)} + G \odot t_0 \odot F_0^{(1)} + F_0^{(1)} \odot t_0^h \odot G^h \right. \\ \left. - \vec{F}_1^{(1)} \odot t_0^{(2)} \odot \vec{F}_1^{(1)} + G \odot \vec{t}_1 \odot \vec{F}_1^{(1)} - \vec{F}_1^{(1)} \odot \vec{t}_1^h \odot G^h \right] \square V_0, \quad (55c)$$

$$t_0^{(2)} = +2i \left[G^h \odot t_0^{(2)} \odot G - F_0^{(2)} \odot t_0^{(1)} \odot F_0^{(2)} + F_0^{(2)} \odot t_0 \odot G + G^h \odot t_0^h \odot F_0^{(2)} \right. \\ \left. - \vec{F}_1^{(2)} \odot t_0^{(1)} \odot \vec{F}_1^{(2)} + \vec{F}_1^{(2)} \odot \vec{t}_1 \odot G - G^h \odot \vec{t}_1^h \odot \vec{F}_1^{(1)} \right] \square V_0. \quad (55d)$$

IV. OPTICAL THEOREM FORMALISM

A. Radiation from a piece of non-equilibrium matter

Optical theorem formalism is an efficient tool to calculate reaction rates including finite particle widths and other in-medium effects [35, 47, 48]. Assume that we deal with a system of a finite size (white body) transparent for radiating quanta. To be specific let us consider anti-neutrino-lepton ($\bar{\nu}l$) production. By the lepton we mean the electron e , muon μ^- , or neutrino ν . We assume that the system is opaque for e and μ^- but transparent for ν and $\bar{\nu}$. Then it is convenient to express all quantities in the Wigner representation doing the transformation (A9) from coordinates $x = X - \xi/2$, $y = X + \xi/2$ to the corresponding Wigner coordinates $(X; q) = (t, \vec{X}; \omega, \vec{q})$. The probability of the anti-neutrino-lepton production can be expressed in terms of the evolution operator S ,

$$\frac{d\mathcal{W}_{X^* \rightarrow \bar{\nu}l}^{\text{tot}}}{d^3x dt d^4q} = \int d\Phi_l (-i\Sigma^{-+}) \delta^{(4)}(q - q_l - q_{\bar{\nu}}), \quad (62)$$

$$-i\Sigma^{-+}(x, q) = \sum_{\{X^*\}} \langle \bar{\nu}l | \delta S^\dagger | X^* \rangle \langle X^* | \delta S | \bar{\nu}l \rangle, \quad (63)$$

where $\delta S = S - 1$, and

$$d\Phi_l = \frac{(1 - n_l) dq_l^3 dq_{\bar{\nu}}^3}{(2\pi)^6 4 \omega_l \omega_{\bar{\nu}}} \quad (64)$$

is the phase-space volume of an antineutrino with the four-momentum $q_{\bar{\nu}} = (\omega_{\bar{\nu}}, \vec{q}_{\bar{\nu}})$ and a lepton with the four-momentum $q_l = (\omega_l, \vec{q}_l)$. If $l = \nu$, the occupation number n_l is to be put zero. The summation goes over the complete set of all possible intermediate states $\{X^*\}$. In Eq. (63) we suppose that electrons or muons can be treated in the quasi-particle approximation. Then there is no need (although possible) to consider them as intermediate states.

Making use of smallness of the weak coupling, we expand the evolution operator as

$$\delta S \approx -i \int_{-\infty}^{+\infty} \mathcal{T} \{ V_W(x_0) S_{\text{nuc}}(x_0) \} dx_0, \quad (65)$$

where V_W is the Hamiltonian of the weak interactions in the interaction representation, S_{nuc} is the part of the S matrix corresponding to strong nuclear interactions, and $\mathcal{T}\{\dots\}$ is the chronological time ordering operator. After substitution of δS into (63) and averaging over a non-equilibrium initial state of the nuclear system, there appear chronologically ordered (G^{--} , F^{--}), anti-chronologically ordered (G^{++} , F^{++}) and disordered (G^{+-} , F^{+-} and G^{-+} , F^{-+}) exact Green's functions.

Once the reaction probability is evaluated according to Eq. (62), the neutrino emissivity in the neutral channel, i.e. with $\bar{\nu}\nu$ production, is given by [35]

$$\epsilon_{\bar{\nu}\nu} = \int \frac{d\mathcal{W}_{X^* \rightarrow \bar{\nu}\nu}^{\text{tot}}}{d^3x dt d^4q} \omega d\omega d^3q. \quad (66)$$

The emissivity in the charged channel, i.e. with $\bar{\nu}l$ production ($l = e^-, \mu^-$), is as follows

$$\epsilon_{\bar{\nu}l} = \int \frac{d\mathcal{W}_{X^* \rightarrow \bar{\nu}l}^{\text{tot}}}{d^3x dt d^4q d^3q_{\bar{\nu}}} \omega_{\bar{\nu}} d^3q_{\bar{\nu}} d^4q. \quad (67)$$

The Lagrangian density for the lepton-nucleon interactions is

$$L = \frac{G}{\sqrt{2}} (j_{\text{ch},\mu} l_{\text{ch}}^\mu + j_{\text{neut},\mu} l_{\text{neut}}^\mu + \text{h.c.}), \quad (68)$$

where $G = 1.166 \times 10^{-5} \text{ GeV}^{-1}$ is the Fermi coupling constant and there are two contributions from charged (ch) and neutral (neut) currents; “h.c.” stands for hermitian conjugated terms. The lepton currents are

$$l_{\text{neut}}^\mu = \bar{\psi}_\nu \gamma^\mu (1 - \gamma_5) \psi_\nu, \\ l_{\text{ch}}^\mu = \bar{\psi}_e \gamma^\mu (1 - \gamma_5) \psi_\nu. \quad (69)$$

The nucleon currents j_{ch} and j_{neut} have vector and axial-vector components

$$j_{\text{ch},\mu} = j_{\mu}^{np}, \quad j_{\text{neut},\mu} = j_{\mu}^{pp} + j_{\mu}^{nn}, \\ j_{\mu}^{np} = \Psi_p^\dagger(p') (J_\mu^V - g_A J_\mu^A) \Psi_n(p), \\ j_{\mu}^{pp} = \frac{1}{2} \Psi_p^\dagger(p') (c_v J_\mu^V - g_A J_\mu^A) \Psi_p(p), \\ j_{\mu}^{nn} = -\frac{1}{2} \Psi_n^\dagger(p') (J_\mu^V - g_A J_\mu^A) \Psi_n(p), \quad (70)$$

where the four vectors $J^{V,A}$ can be written for non-relativistic nucleons as, cf. Ref. [123],

$$J^V = (1, \frac{\vec{p}' + \vec{p}}{2m_N}), \quad J^A = (\frac{\vec{\sigma}(\vec{p}' + \vec{p})}{2m_N}, \vec{\sigma}), \quad (71)$$

and $c_v = 1 - 4 \sin^2 \theta_W \simeq 0.08$, θ_W is the Weinberg angle; $g_A \simeq 1.26$ is the axial-vector coupling constant, $p = (p_0, \vec{p})$, \vec{p} and \vec{p}' are momenta of incoming and outgoing nucleons. Compared to the frequently used expression, that includes only leading terms in the non-relativistic expansion, e.g., see Ref. [51], we following [94, 95] retain here sub-leading terms $\propto v_F$. Although in many cases the terms $\propto v_F$ yield small corrections to leading-order results, in some particular cases the leading-term contribution may vanish because of symmetry constraints and then sub-leading terms become dominant. Such an example will be studied below. The bare current vertex involves the bare nucleon mass, see Eq. (71). In medium, however, nucleon wave functions are normalized to one quasi-particle rather than to one free particle, provided the nucleons are treated in the quasi-particle approximation. Hence, the bare nucleon mass m_N is to be replaced by the in-medium nucleon mass m_N^* .

The structure of the weak-interaction Lagrangian (68) suggests that we can detach leptonic currents

$$\Sigma^{-+} = \frac{G^2}{2} \Sigma_{\text{nuc},\mu\nu}^{-+} \sum_{\text{spin}} \{ l^\mu l^\nu \}, \quad (72)$$

and deal with the object determined by the strong interactions only

$$\Sigma_{\text{nuc},\mu\nu}^{-+}(X, q) = \int d^4\xi e^{iq\xi} i \langle j_\nu^\dagger(X - \xi/2) j_\mu(X + \xi/2) \rangle. \quad (73)$$

$\Sigma_{\text{nuc},\mu\nu}^{-+}$ is the full $(-+)$ self-energy for the nuclear processes; the current j stands here for charged or neutral nucleon currents defined in (70). Quantum states and operators are taken in the Heisenberg picture. The sum in Eq. (72) runs over lepton spins.

In the graphical form, the general expression for the probability of the lepton (electron, muon, neutrino) and anti-neutrino production is depicted as

$$\frac{G^2}{2} i \Sigma_{\text{nuc},\mu\nu}^{-+} \sum_{\text{spin}} \{l^\mu l^\nu\} = \begin{array}{c} l \\ \nearrow \\ \bar{\nu} \end{array} \begin{array}{c} + \\ \text{hatched block} \\ - \end{array} \begin{array}{c} \bar{\nu} \\ \nwarrow \\ l \end{array}. \quad (74)$$

The hatched block has the meaning of the $(-+)$ self-energy of the virtual Z or W bosons which convert in a lepton l and an anti-neutrino $\bar{\nu}$. The block determines the gain term in the generalized kinetic equation for the full $(-+)$ Green's functions of the virtual boson. If one integrates over e/μ (closes the e/μ line in diagrams), one recovers the gain term for the $\bar{\nu}$ in the charged current processes, see Refs. [47, 107, 124]. This circumstance allows to use the given method in different non-equilibrium problems, like in description of the neutrino/antineutrino transport. Note that the generalized kinetic equation for $\bar{\nu}$ is greatly simplified, if conditions for the quasi-particle approximation are fulfilled, see [48, 107, 125].

The integration over the lepton phase-space can be performed separately from the calculation of $\Sigma_{\text{nuc},\mu\nu}^{-+}$. If we introduce the leptonic tensor

$$T_{\text{lept}}^{\mu\nu}(q) = \int d\Phi_l \sum_{\text{spin}} \{l^\mu l^\nu\} \delta^{(4)}(q - q_l - q_{\bar{\nu}}), \quad (75)$$

the reaction probability (62) takes the form

$$\frac{d\mathcal{W}_{X^* \rightarrow \bar{\nu}l}^{\text{tot}}}{d^3x dt d^4q} = -i \frac{G^2}{2} \Sigma_{\text{nuc},\mu\nu}^{-+}(X, q) T_{\text{lept}}^{\mu\nu}(q). \quad (76)$$

For evaluation of the emissivity in Eq. (67) we also need the following expression

$$\int \frac{d\mathcal{W}_{X^* \rightarrow \bar{\nu}l}^{\text{tot}}}{d^3x dt d^4q d^3q_{\bar{\nu}}} \omega_{\bar{\nu}} d^3q_{\bar{\nu}} = -i \frac{G^2}{2} \Sigma_{\text{nuc},\mu\nu}^{-+}(X, q) \times \tilde{T}_{\text{lept}}^{\mu\nu}(q), \quad (77)$$

where $l = e^-, \mu^-$ and

$$\tilde{T}_{\text{lept}}^{\mu\nu}(q) = \int d\Phi_l \omega_{\bar{\nu}} \sum_{\text{spin}} \{l^\mu l^\nu\} \delta^{(4)}(q - q_l - q_{\bar{\nu}}). \quad (78)$$

The tensors $T_{\text{lept}}^{\mu\nu}$ and $\tilde{T}_{\text{lept}}^{\mu\nu}$ are calculated in Appendix B. Note that the self-energies, Σ_{nuc}^{-+} , in Eqs. (76) and (77)

are to be constructed with the neutral and charged nucleon currents from Eq. (70), respectively.

As we have discussed in the Introduction, the standard calculation of the reaction rates is done with the help of summation of the squared matrix elements of reactions, see [90]. This is fully correct procedure, if one treats the processes perturbatively, i.e. provided there is a small expansion parameter. One nucleon processes are related to perturbative Σ^{-+} diagrams with only one the nucleon G_0^{-+} Green's function in expansion of (74), two-nucleon processes are related to the diagrams with two nucleon G_0^{-+} Green functions, etc. However, this procedure fails when applied to strongly interacting systems. The description of even a one-nucleon process includes infinite number of perturbative diagrams with the NN -interactions, since the coupling is not small. Nevertheless, one is able to separate processes using the quasi-particle approximation provided excitation energies are sufficiently low $\omega \ll \epsilon_F$ (when the fermion width is small). Then, diagrams with one quasi-particle nucleon G^{-+}/F^{-+} Green's function describe the one-nucleon reactions, with two G^{-+}/F^{-+} Green's functions describe two-nucleon reactions, etc. The calculations of the reaction rates based on application of the optic theorem formalism and calculations using the ordinary formalism of computing squared reaction-matrix elements yield the same results [35, 46]. In general case, when particle widths are not small, situation becomes much more involved. Then calculations using the squared matrix elements become invalid, and the only possibility to calculate the emissivity from the piece of matter is to use the closed-diagram technique. Below we formulate a general method and then discuss the quasi-particle approximation.

B. Diagrammatic decomposition in terms of full $(-+)$ Green functions

1. Fermions with finite width

The hatched block in Eq. (74) is the sum of all closed diagrams written in terms of full Green's functions. External $(-+)$ signs mean that each diagram in the series contains at least one $(-+)$ nucleon Green's function (G^{-+} and additionally F^{-+} , for a system with pairing). The latter function is especially important. It obeys the Kadanoff-Baym kinetic equation. Various contributions from $\{X\}$ can be classified according to the number N of exact G^{-+}/F^{-+} nucleon Green's functions (lines in the diagram):

$$\frac{d\mathcal{W}_{X^* \rightarrow \bar{\nu}l}^{\text{tot}}}{d^3x dt} = \int d\Phi_l \delta^{(4)}(q - q_l - q_{\bar{\nu}}) \times \left[\begin{array}{c} l \\ \nearrow \\ \bar{\nu} \end{array} \begin{array}{c} + \\ \text{hatched block} \\ - \end{array} \begin{array}{c} \bar{\nu} \\ \nwarrow \\ l \end{array} \begin{array}{c} + \\ \text{hatched block} \\ - \end{array} \begin{array}{c} l \\ \nearrow \\ \bar{\nu} \end{array} \begin{array}{c} + \\ \text{hatched block} \\ - \end{array} \begin{array}{c} \bar{\nu} \\ \nwarrow \\ l \end{array} + \dots \right]. \quad (79)$$

The quasi-particle approximation for fermions can be utilized only if the energy of radiating quanta ω is larger than the nucleon width ($\omega \sim \epsilon^* \sim T \gg \Gamma_N$), i.e., inequality $\omega \sim T \ll \epsilon_F$ must hold, see Refs. [35, 47]. In this case the contributions of various processes encoded in a closed diagram can be made visible by cutting the diagrams through the $(+-)$ and $(-+)$ lines. The cut means taking off the energy integral provided the spectral functions of fermionic quasi-particles can be reduced to the δ -functions. This restricts the fermion energy to an in-medium mass shell. The $N = 1$ term describes the DU process, and $N = 2$, the MMU and MNB processes.

In general case, when the fermion width cannot be neglected, the cut through the $(-+)$, $(+-)$ lines has only a symbolic meaning. Nevertheless, the separation of the diagrams according to the number of the full $(-+)$ Green's functions proves to be helpful also in this case [47]. Note that now each diagram in (79) represents a whole class of perturbative diagrams of any order in the interaction strength and in the number of loops.

The full set of topologically distinct skeleton diagrams for Σ^{--} written in terms of full $(-+)$ Green's functions can be explicitly presented as a series in N [47]. For $N = 1$ and $N = 2$ we have

$$^{+-}N=1^{--} = ^{+-} \text{ (circle) } ^{--}, \quad (80)$$

$$\begin{aligned} ^{+-}N=2^{--} &= ^{+-} \text{ (two loops) } ^{--} + ^{+-} \text{ (two loops with dashed line) } ^{--} \\ &+ ^{+-} \text{ (two loops with dashed line) } ^{--} + \dots \end{aligned} \quad (81)$$

For $N = 3$ we have, for example, contributions of the type

$$\begin{aligned} ^{+-}N=3^{--} &= ^{+-} \text{ (three loops) } ^{--} \\ &+ ^{+-} \text{ (three loops with dashed line) } ^{--} + \dots \end{aligned} \quad (82)$$

Note that for $N \geq 3$ there appear multi-cut diagrams (see the last explicitly presented diagram in (82)). The NN interaction block in Eqs. (80,81) and (82) is the full block containing the vertices of one particular sign, e.g.,

$$^{--} \text{ (wavy line) } ^{--} = ^{--} \text{ (wavy line) } ^{--} + ^{--} \text{ (wavy line) } ^{--} + \dots \quad (83)$$

and the analogous equation for the $(++)$ block. Since only the same-sign vertices are permitted in Eq. (83), no (\pm, \mp) [i.e., $(+, -)$ and $(-, +)$] lines appear in these diagrams. The thick double-wavy lines stand here for the exact boson $(--)$ Green's functions or an iterated two-body potentials:

$$^{--} \text{ (wavy line) } ^{--} = \text{ (diagram with dot) } + \sum_{k,l} \text{ (diagram with dot) } ^{--}. \quad (84)$$

The full dot in the vertex is the renormalized in-medium vertex, which includes all diagrams with one sign, i.e. it is irreducible with respect to the full $(\pm\mp)$ nucleon Green's functions. This means that it contains only $(--)$ or $(++)$ Green's functions. We denote such vertices as τ^- and τ^+ , e.g.,

$$\text{ (diagram with dot) } ^{--} = \text{ (diagram with dot) } ^{--} + \text{ (diagram with dot) } ^{--} + \text{ (diagram with dot) } ^{--}. \quad (85)$$

Here we assume that the bare vertex is time-local, i.e. it carries on only one Keldysh index instead of three indices in general case. Note that full one-sign Green's functions entering Eqs. (83,84,85), would contain alternative sign diagrams, if they were expanded in perturbative series with respect to the bare Green's functions. To simplify discussion, we do not include in Eq. (85) the processes when the weak interaction (dashed line) is coupled directly to the intermediate pion line due to the $\pi\pi \rightarrow \bar{\nu}l$ processes, see the first diagram in Eq. (90) below.

We did not show the direction of fermion lines in the diagrams, since it can be picked out at will in closed-loop diagrams. Once some direction is chosen, the arrows in the diagrams (83,84,85) follow.

For a theory of fermions interacting with bosons the contribution with the fewest number of external particles is three (rather than four as for processes described with the Boltzmann kinetics). Indeed, the cut through the one-loop diagram in Eq. (80) shows that in dense matter an off-shell fermion can decay into a fermion plus a boson *vice versa*. For these processes it is important that all particles have a finite width in dense matter. The formation and decay processes which are forbidden by the energy-momentum conservation in the free space, can occur in the dense matter without principal restrictions. Therefore the most important term in the series (79) is the first (one-loop) diagram (80), which is positively definite, and corresponds to the first term of the classical Langevin equation, for details see [47].

The classical Langevin process deals with the propagation of a single charge (say a proton) in a neutral medium (e.g in the neutron matter). Therefore for this case only those diagrams occur, where both photon vertices attach to the same proton line. In the quasiclassical limit for fermions (with small fermion occupations $n_f \ll 1$; in case of equilibrium matter $n_f(p_0) = [\exp((p_0 - \mu)/T) + 1]^{-1}$)

all the diagrams

$$-i\Sigma_{cl}^{-+} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} \quad (86)$$

with arbitrary number of $(-+)$ lines are summed up leading to the diffusion result, see Ref. [47] for detailed discussion. Each of these diagrams with the n vertical insertions corresponds to the n -th term in the Langevin process, where hard scatterings occur at random with a constant mean collision rate $\Gamma_f = -2\Im\Sigma_f^R$.

For particle propagation in an external field, e.g., for the scattering on infinitely heavy centers (proper Landau-Pomeranchuk-Migdal effect), only the one-loop diagram remains, where the fermion line is given by

$$\text{solid line} = \text{solid line} + \text{solid line with vertical dashed line and star} \quad (87)$$

since one deals then with a genuine one-body problem.

In the general case of a non-equilibrium system all above equations, being presented in the Wigner representation, are very cumbersome. To simplify the problem one may use the gradient expansion in the convolutions of two-point functions, see Eq. (A11). In general case, the Wigner transformation will produce an infinite tower of nested gradient terms. Assuming that a piece of non-equilibrium matter under consideration evolves very slowly in time and space, one may keep only first-order gradient terms. First-order gradient terms in the expansion of the collision term $C_{col} = \Sigma^{-+}G^{+-} - \Sigma^{+-}G^{-+}$ are attributed to the memory effects in the generalized kinetic equation [107]. In the standard derivation of the kinetic Kadanoff-Baym equation one simplifying usually drops these effects [19, 112]. As has been shown in [107] the memory terms are of the same gradient order as other terms in the Kadanoff-Baym equation and should be kept, because of the local part of the collision term is also of the first gradient order (since C_{col} being zero in the thermal equilibrium state). However, in the given paper we are interested only in the calculation of the production rates in direct reactions from a piece of the non-equilibrium matter, which are fully determined by the quantity Σ^{-+} . Since $\Sigma^{-+} \neq 0$ in the thermal equilibrium, the gradient corrections to it are small and can be neglected provided the given non-equilibrium piece of matter evolves very slowly in time and space, that we further assume. Therefore, in further we will calculate only the local part of the Σ^{-+} term.

2. Quasiparticle approximation for fermions

The one-loop diagram (80) calculated with the quasiparticle fermion propagators determines the one-nucleon reactions: the DU reactions and the PBF (in case of the superfluid matter) [35, 38, 68]. The contribution to the DU process vanishes for $n < n_c^{DU}$.

The two-nucleon processes are encoded in the $N = 2$ term in Eq. (79) and are given in the quasi-particle approximation by the diagrams

$$\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} \quad (88)$$

Note that the first diagram in (88) is not allowed in terms of the full Green's functions with the width (compare with (81)) but it should be explicitly presented in the quasi-particle picture, see [35, 47]. After the cut over $(-+)$, $(+-)$ lines the diagrams (88) are separated by two pieces and correspond to the processes

$$\text{diagram 1} , \text{diagram 2} \quad (89)$$

shown here for the paired potential interaction.

In case when the NN interaction amplitude is mainly controlled by the soft pion exchange in the reaction channel under consideration (for $n \gtrsim n_0$), instead of (89) one has

$$\text{diagram 1} , \text{diagram 2} , \text{diagram 3} , \text{diagram 4} \quad (90)$$

Ref. [46] calculated the rate of MMU and MNB processes taking into account in-medium effects for the case of non-superfluid nucleon matter. Evaluations [28, 38, 46] have shown that the dominating contribution to MMU rate comes from the first two diagrams of the series (90), whereas the third diagram, which naturally generalizes the corresponding MU (FOPE) contribution, gives only a small correction for $n \gtrsim n_0$. As is seen from comparison of Eqs. (89) and (90), the first diagram (90) is absent,

if one approximates the nucleon-nucleon interaction by a two-body potential.

The diagrams that can be cut into more than two pieces (e.g., see the last explicitly presented diagram in Eq. (82)) are proportional to powers of independent $(L^{-+})^{2+n}(L^{+-})^m$ loops, m, n are positive integer numbers, whereas the diagrams for a two-nucleon process have only two L^{-+} loops, and they decay after the cut into two parts.

In Ref. [47] it was shown how for the processes with the radiation of soft quanta one can simply incorporate the effects of a finite fermion width into the results calculated within the quasi-particle approximation for fermions (i.e., the fermion width $\Gamma_f = -2\Im\Sigma_f^R$ is put to zero). For this purpose it is sufficient to multiply the quasi-particle result by a pre-factor. For example, comparing one-loop result at a non-zero value of the nucleon width Γ_f with the first non-zero diagram in the quasi-particle approximation (when one puts $\Gamma_f \rightarrow 0$ in the expressions for the Green's functions) one gets

$$\begin{array}{c} \text{diagram} \\ \pm \end{array} = C_0(\omega) \left\{ \begin{array}{c} \text{diagram} \\ \pm \end{array} \right\}_{\text{QP}} \quad (91)$$

QP means here the quasi-particle result. At a small momentum \vec{q} of the radiated quantum the correction factor is equal to

$$C_0(\omega) = \frac{\omega^2}{\omega^2 + \Gamma_N^2}, \quad (92)$$

which removes the singularity of the quasi-particle production rate for small ω . This factor complies with the replacement $\omega \rightarrow \omega + i\Gamma_N$ in the retarded Green's function. Correction factors for the higher order diagrams also can be derived. Here we quote corresponding results for the next lowest order diagrams:

$$\begin{array}{c} \text{diagram} \\ \pm \end{array} = C_1(\omega) \left\{ \begin{array}{c} \text{diagram} \\ \pm \end{array} \right\}_{\text{QP}} \quad (93)$$

$$\begin{array}{c} \text{diagram} \\ \pm \end{array} = C_0(\omega) \left\{ \begin{array}{c} \text{diagram} \\ \pm \end{array} \right\}_{\text{QP}} \quad (94)$$

with $C_0(\omega)$ from (92) and

$$C_1(\omega) = \omega^2 \frac{\omega^2 - \Gamma_N^2}{(\omega^2 + \Gamma_N^2)^2}. \quad (95)$$

In case $T \ll \epsilon_F$ for typical energy of the radiation $\omega \sim T$ one has $\omega \gg \Gamma_N$, since $\Gamma_N \propto T^2/\epsilon_F$, and thus $C_0 \simeq C_1 \simeq 1$, see [47]. In general case the full radiation rate is obtained by summation of all diagrams in the series (79).

C. Resummation of the two-fermion interaction out of equilibrium. Bosonization of the interaction

In this section we work out resummation of the two-fermion interaction amplitude starting from a bare interaction, which is local in time but not necessarily local in space. This is the generalization of the procedure performed in Ref. [47] for the point-like interaction, local both in space and in time. We construct the compact expression for Σ^{-+} self-energy in equilibrium and non-equilibrium cases and illustrate how the decomposition with respect to the number of $(+-)$ and $(-+)$ lines works. In order to simplify the consideration we first study the normal matter and then perform generalizations to the superfluid matter.

Consider the particle-hole channel with the full two-fermion interaction amplitude determined by

$$\begin{array}{c} \text{diagram} \\ i \end{array} j = \begin{array}{c} \text{diagram} \\ i \end{array} j + \sum_{k,l} \begin{array}{c} \text{diagram} \\ i \end{array} k \begin{array}{c} \text{diagram} \\ l \end{array} j \quad (96)$$

with some particle-hole irreducible bare time-local interaction \mathcal{G}_0 . Without the first-order gradient terms included, the diagrams in Eq. (96) correspond to the following expression in the Wigner representation

$$\begin{aligned} \mathcal{G}^{ji}(p', p; q) &= \mathcal{G}_0^{ji}(p', p) + \sum_{kl} \int \frac{dp''^4}{(2\pi)^4} i \\ &\times \mathcal{G}^{jl}(p', p''; q) G^{lk}(p'' + \frac{q}{2}) G^{kl}(p'' - \frac{q}{2}) \mathcal{G}_0^{ki}(p'', p; q) \end{aligned} \quad (97)$$

where each element is to be understood as depending additionally on the Wigner's X variable. Since in the local approximation exploited here this variable will enter all expressions as a common parameter, we will not write it explicitly in the expressions below. In Eq. (97), p, p' and p'' are the relative momenta in incoming, outgoing and intermediate channels, respectively; q is the exchanged momentum in the particle-hole channel. Note that in case under consideration the bare interaction is diagonal in the Schwinger-Keldysh space, i.e

$$\mathcal{G}_0^{\pm\mp} = 0, \quad \mathcal{G}_0^{--} = -\mathcal{G}_0^{++} = \mathcal{G}_0. \quad (98)$$

We introduce the products of the Green's functions

$$\tilde{L}^{jk}(p; q) = -i G^{jk}(p + q/2) G^{kj}(p - q/2), \quad (99)$$

which are related to the bare self-energies as

$$\Sigma_0^{ij} = \int \frac{dp^4}{(2\pi)^4} \tau_0^i(p; q) \tilde{L}^{ij}(p; q) \tau_0^j(p; q). \quad (100)$$

Since here the bare vertices are assumed to be time local, they carry only one Keldysh index, $\tau_k^{ij} = \tau^i \delta_{ij} \delta_{ik}$. For the vertex independent on the fermion momentum p the self-energy reads

$$\Sigma_0^{ij} = \tau_0^2 L^{ij}(q), \quad L^{ij}(q) = \int \frac{dp^4}{(2\pi)^4} \tilde{L}^{ij}(p; q), \quad (101)$$

where we introduced the loop-functions L^{ij} .

If we formally extend the products (99) as

$$\tilde{L}^{jk}(p; q) \rightarrow \tilde{L}^{jk}(p, p'; q) = \tilde{L}^{jk}(p; q)(2\pi)^4 \delta^{(4)}(p - p'),$$

Eq. (96) can be presented as

$$\mathcal{G}^{ji}(p', p; q) = \mathcal{G}_0^{ji}(p', p) + \sum_{kl} \int \frac{dp''^4}{(2\pi)^4 i} \int \frac{dp'''^4}{(2\pi)^4} \times \mathcal{G}^{jl}(p', p''; q) \tilde{L}^{lk}(p'', p'''; q) \mathcal{G}_0^{ki}(p''', p; q). \quad (102)$$

The integral equation (102) can be interpreted as a matrix equation in the discretized momentum space [126]. The integration turns into a summation over the grid points with the appropriate weights. Then in terms of matrices \mathcal{G} , \mathcal{G}_0 and \tilde{L} , Eq. (96) takes the form

$$\mathcal{G}^{ji} = \mathcal{G}_0^{ji} + \sum_{kl} \mathcal{G}^{jl} \cdot \tilde{L}^{lk} \cdot \mathcal{G}_0^{ki}, \quad (103)$$

where dot products emphasize the matrix multiplication. For example, the bare self-energy reads in these notations as $\Sigma_0^{ij} = \tau_0^i \cdot \tilde{L}^{ij} \cdot \tau_0^j$. Working with matrices we can proceed with the solution of Eq. (103) which we rewrite as

$$\begin{aligned} \mathcal{G}^{--} &= \mathcal{G}_0^{--} + \mathcal{G}^{--} \cdot \tilde{L}^{--} \cdot \mathcal{G}_0^{--} + \mathcal{G}^{+-} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_0^{--}, \\ \mathcal{G}^{++} &= \mathcal{G}_0^{++} + \mathcal{G}^{++} \cdot \tilde{L}^{++} \cdot \mathcal{G}_0^{++} + \mathcal{G}^{+-} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_0^{++}, \\ \mathcal{G}^{+-} &= \mathcal{G}^{+-} \cdot \tilde{L}^{--} \cdot \mathcal{G}_0^{--} + \mathcal{G}^{++} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_0^{--}, \\ \mathcal{G}^{-+} &= \mathcal{G}^{-+} \cdot \tilde{L}^{++} \cdot \mathcal{G}_0^{++} + \mathcal{G}^{--} \cdot \tilde{L}^{-+} \cdot \mathcal{G}_0^{++}. \end{aligned} \quad (104)$$

Introducing the quantity, called in [47] the residual interaction,

$$\begin{aligned} \mathcal{G}_{\text{res}}^{\pm\pm} &= \mathcal{G}_0^{\pm\pm} \cdot [1 - \tilde{L}^{\pm\pm} \cdot \mathcal{G}_0^{\pm\pm}]^{-1} \\ &= [1 - \mathcal{G}_0^{\pm\pm} \cdot \tilde{L}^{\pm\pm}]^{-1} \cdot \mathcal{G}_0^{\pm\pm}, \end{aligned} \quad (105)$$

we rewrite the above set of equations as

$$\begin{aligned} \mathcal{G}^{--} &= \mathcal{G}_{\text{res}}^{--} + \mathcal{G}^{+-} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_{\text{res}}^{--}, \\ \mathcal{G}^{++} &= \mathcal{G}_{\text{res}}^{++} + \mathcal{G}^{+-} \cdot \tilde{L}^{-+} \cdot \mathcal{G}_{\text{res}}^{++}, \\ \mathcal{G}^{+-} &= \mathcal{G}^{+-} \cdot \tilde{L}^{--} \cdot \mathcal{G}_{\text{res}}^{--}, \\ \mathcal{G}^{-+} &= \mathcal{G}^{-+} \cdot \tilde{L}^{++} \cdot \mathcal{G}_{\text{res}}^{++}. \end{aligned}$$

Substituting \mathcal{G}^{+-} , \mathcal{G}^{-+} from the last two equations into the first two equations and using the notations

$$\begin{aligned} Z_r^{\mp\mp} &= [1 - \tilde{L}^{\mp\pm} \cdot \mathcal{G}_{\text{res}}^{\pm\pm} \cdot \tilde{L}^{\pm\mp} \cdot \mathcal{G}_{\text{res}}^{\mp\mp}]^{-1}, \\ Z_l^{\mp\mp} &= [1 - \mathcal{G}_{\text{res}}^{\mp\mp} \cdot \tilde{L}^{\mp\pm} \cdot \mathcal{G}_{\text{res}}^{\pm\pm} \cdot \tilde{L}^{\pm\mp}]^{-1} \end{aligned} \quad (106)$$

we arrive at the formal solution of Eq. (103):

$$\mathcal{G}^{--} = \mathcal{G}_{\text{res}}^{--} \cdot Z_r^{--} = Z_l^{--} \cdot \mathcal{G}_{\text{res}}^{--}, \quad (107a)$$

$$\mathcal{G}^{++} = \mathcal{G}_{\text{res}}^{++} \cdot Z_r^{++} = Z_l^{++} \cdot \mathcal{G}_{\text{res}}^{++}, \quad (107b)$$

$$\begin{aligned} \mathcal{G}^{+-} &= \mathcal{G}_{\text{res}}^{+-} \cdot Z_r^{+-} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_{\text{res}}^{--} \\ &= \mathcal{G}_{\text{res}}^{++} \cdot \tilde{L}^{+-} \cdot Z_l^{--} \cdot \mathcal{G}_{\text{res}}^{--}, \end{aligned} \quad (107c)$$

$$\begin{aligned} \mathcal{G}^{-+} &= \mathcal{G}_{\text{res}}^{--} \cdot Z_r^{--} \cdot \tilde{L}^{-+} \cdot \mathcal{G}_{\text{res}}^{++} \\ &= \mathcal{G}_{\text{res}}^{--} \cdot \tilde{L}^{-+} \cdot Z_l^{++} \cdot \mathcal{G}_{\text{res}}^{++}. \end{aligned} \quad (107d)$$

This solution for the bosonized interaction describes propagation of effective boson, such as phonon, plasmon etc., in non-equilibrium systems. These effective bosons can be treated on the same footing as all other effective quanta. The phase space distribution of such bosons is given by the \mathcal{G}^{+-} and \mathcal{G}^{-+} Wigner densities.

Eq. (8) for the retarded Green's function decouples from the other Eqs. (7). Let us demonstrate that the same occurs for the resummed interaction (107). We define the quantity

$$\tilde{L}^R \equiv \tilde{L}^{--} - \tilde{L}^{-+}, \quad (108)$$

which, being integrated over 4-momentum p gives the retarded loop-function $L^R = L^{--} - L^{-+}$ as follows from (A6). Similarly we define

$$\tilde{L}^A = \tilde{L}^{--} - \tilde{L}^{+-}. \quad (109)$$

Using relations (A12) we are able to prove that these quantities are connected by the relation $[\tilde{L}^R]^\dagger = \tilde{L}^A$ like the retarded and advanced Green's functions and self-energies. For the case of an energy-independent bare interaction \mathcal{G}_0 , Eqs. (A13, A14, A15) imply the useful relation

$$\mathcal{G}_0 \cdot (\tilde{L}^{--} - \tilde{L}^{-+}) \cdot \mathcal{G}_0 = \mathcal{G}_0 \cdot (\tilde{L}^{+-} - \tilde{L}^{++}) \cdot \mathcal{G}_0. \quad (110)$$

With the help of this relation we obtain from Eqs. (104) and Eq. (98):

$$\begin{aligned} \mathcal{G}^{--} + \mathcal{G}^{-+} &= \mathcal{G}_0 + (\mathcal{G}^{--} + \mathcal{G}^{-+}) \cdot \tilde{L}^R \cdot \mathcal{G}_0, \\ -\mathcal{G}^{+-} - \mathcal{G}^{++} &= \mathcal{G}_0 - (\mathcal{G}^{+-} + \mathcal{G}^{++}) \cdot \tilde{L}^R \cdot \mathcal{G}_0, \end{aligned}$$

where we used that $(\mathcal{G}^{--} \cdot \tilde{L}^{--} - \mathcal{G}^{-+} \cdot \tilde{L}^{++} + \mathcal{G}^{-+} \cdot \tilde{L}^{+-} - \mathcal{G}^{--} \cdot \tilde{L}^{-+}) \cdot \mathcal{G}_0 = (\mathcal{G}^{--} + \mathcal{G}^{-+}) \cdot \tilde{L}^R \cdot \mathcal{G}_0$ and analogously $(\mathcal{G}^{++} \cdot \tilde{L}^{++} + \mathcal{G}^{+-} \cdot \tilde{L}^{-+} + \mathcal{G}^{+-} \cdot \tilde{L}^{--} + \mathcal{G}^{++} \cdot \tilde{L}^{+-}) \cdot \mathcal{G}_0 = -(\mathcal{G}^{++} + \mathcal{G}^{+-}) \cdot \tilde{L}^R \cdot \mathcal{G}_0$. Thus, we can introduce the retarded interaction amplitude

$$\begin{aligned} \mathcal{G}^R &= \mathcal{G}^{--} + \mathcal{G}^{-+} = -\mathcal{G}^{+-} - \mathcal{G}^{++} \\ &= \mathcal{G}_0 \cdot [1 - \tilde{L}^R \cdot \mathcal{G}_0]^{-1} = [1 - \mathcal{G}_0 \cdot \tilde{L}^R]^{-1} \cdot \mathcal{G}_0, \end{aligned} \quad (111)$$

expressed only through the quantity \tilde{L}^R , which convolution with \mathcal{G}_0 , like $\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0$ possesses the retarded properties.

D. Physical meaning of multi-piece diagrams

In general case the total radiation rate is obtained by summation of all diagrams in (79). Some of the diagrams shown, e.g., in the second line in Eq. (82) give more than two pieces, if being cut. Therefore, they do not reduce to the Feynman amplitudes. The role of these diagrams will be illustrated in the given sub-section.

1. Non-equilibrium systems

Consider the RPA-like set of the self-energy diagrams

$$-i\Sigma_{\text{RPA}}^{ij} = -i \text{ (diagram with two loops)} + \sum_{k,l} -i \text{ (diagram with two loops and a wavy line)}. \quad (112)$$

Note that this is only a RPA subset of all possible self-energy diagrams. In the Wigner representation with the omitted gradient terms Eq. (112) reads

$$\Sigma_{\text{RPA}}^{ij} = \tau_0^i \cdot \tilde{L}^{ij} \cdot \tau_0^j + \sum_{kl} \tau_0^i \cdot \tilde{L}^{ik} \cdot \mathcal{G}^{kl} \cdot \tilde{L}^{lj} \cdot \tau_0^j. \quad (113)$$

According to Eq. (112) the quantity Σ_{RPA}^{++} , which determines the production probability, includes now the following terms

$$\begin{aligned} & \text{(diagram with two loops)} + \text{(diagram with two loops and a shaded circle)} + \text{(diagram with two loops and a shaded circle)} + \dots \\ & \text{(diagram with two loops and a shaded circle)} + \text{(diagram with two loops and a shaded circle)} + \dots \\ & \text{(diagram with two loops and a shaded circle)} + \text{(diagram with two loops and a shaded circle)} + \dots \end{aligned} \quad (114)$$

We see that if one singles out infinite tower of terms with only one $(-+)$ loop, their summation will lead to the renormalization of left and right vertices in the one-loop diagram, see the $N = 1$ term in Eq. (80). In addition Eq. (113) contains terms with many repeated $(+-)$ and $(-+)$ loops, i.e. the multi-piece diagrams. However, the RPA series does not include many other terms with $N \geq 2$, as it is seen from comparison with Eq. (82).

From Eq. (113) we write now

$$\begin{aligned} \Sigma_{\text{RPA}}^{++} &= \tau_0^- \cdot \tilde{L}^{-+} \cdot \tau_0^+ \\ &+ \tau_0^- \cdot \tilde{L}^{--} \cdot \mathcal{G}^{--} \cdot \tilde{L}^{-+} \cdot \tau_0^+ \\ &+ \tau_0^- \cdot \tilde{L}^{-+} \cdot \mathcal{G}^{++} \cdot \tilde{L}^{++} \cdot \tau_0^+ \\ &+ \tau_0^- \cdot \tilde{L}^{-+} \cdot \mathcal{G}^{+-} \cdot \tilde{L}^{-+} \cdot \tau_0^+ \\ &+ \tau_0^- \cdot \tilde{L}^{--} \cdot \mathcal{G}^{+-} \cdot \tilde{L}^{++} \cdot \tau_0^+. \end{aligned} \quad (115)$$

Using another self-energy

$$\begin{aligned} \Sigma_{\text{RPA}}^{--} &= \tau_0^- \cdot \tilde{L}^{--} \cdot \tau_0^- \\ &+ \tau_0^- \cdot \tilde{L}^{--} \cdot \mathcal{G}^{--} \cdot \tilde{L}^{--} \cdot \tau_0^- \\ &+ \tau_0^- \cdot \tilde{L}^{-+} \cdot \mathcal{G}^{++} \cdot \tilde{L}^{+-} \cdot \tau_0^- \\ &+ \tau_0^- \cdot \tilde{L}^{-+} \cdot \mathcal{G}^{+-} \cdot \tilde{L}^{--} \cdot \tau_0^- \\ &+ \tau_0^- \cdot \tilde{L}^{--} \cdot \mathcal{G}^{+-} \cdot \tilde{L}^{+-} \cdot \tau_0^-, \end{aligned} \quad (116)$$

we can determine the retarded combination $\Sigma_{\text{RPA}}^R = \Sigma_{\text{RPA}}^{--} + \Sigma_{\text{RPA}}^{++}$. The direct calculation yields

$$\begin{aligned} \Sigma_{\text{RPA}}^{--} + \Sigma_{\text{RPA}}^{++} &= \tau_0^- \cdot \tilde{L}^R \cdot \tau_0^- \\ &+ \tau_0^- \cdot \tilde{L}^R \cdot (\mathcal{G}^{--} + \mathcal{G}^{++}) \cdot \tilde{L}^R \cdot \tau_0^- \\ &+ \tau_0^- \cdot \tilde{L}^R \cdot (\mathcal{G}^{--} + \mathcal{G}^{++} + \mathcal{G}^{+-} + \mathcal{G}^{-+}) \cdot \tilde{L}^R \cdot \tau_0^-. \end{aligned}$$

Taking into account Eq. (111) we obtain

$$\begin{aligned} \Sigma_{\text{RPA}}^R &= \Sigma_{\text{RPA}}^{--} + \Sigma_{\text{RPA}}^{++} \\ &= \tau_0^- \cdot \tilde{L}^R \cdot \tau_0^- + \tau_0^- \cdot \tilde{L}^R \cdot \mathcal{G}^R \cdot \tilde{L}^R \cdot \tau_0^- \\ &= \tau_0^- \cdot \tilde{L}^R \cdot [1 - \mathcal{G}_0 \cdot \tilde{L}^R]^{-1} \cdot \tau_0^-. \end{aligned} \quad (117)$$

Equations (111) and (117) express the retarded interaction and the retarded self-energy through the bare interaction \mathcal{G}_0 and the quantity \tilde{L}^R determined by Eq. (108).

It is possible to present expression for Σ_{RPA}^{++} in a more compact form. Using relations

$$\tilde{L}^{\pm\pm} \cdot \mathcal{G}_{\text{res}}^{\pm\pm} = [1 - L^{\pm\pm} \cdot \mathcal{G}_0^{\pm\pm}]^{-1} - 1, \quad (118a)$$

$$\begin{aligned} \tilde{L}^{-+} \cdot \mathcal{G}_{\text{res}}^{++} \cdot Z_r^{++} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_{\text{res}}^{--} \cdot \tilde{L}^{-+} \\ = \tilde{L}^{-+} \cdot (Z_l^{++} - 1), \end{aligned} \quad (118b)$$

$$\tilde{L}^{-+} \cdot Z_l^{++} = Z_r^{--} \cdot \tilde{L}^{-+}, \quad (118c)$$

we rewrite Eq. (115) as

$$\begin{aligned} \Sigma_{\text{RPA}}^{++} &= \tau_{\text{res}}^- \cdot Z_r^{--} \cdot \tilde{L}^{-+} \cdot \tau_{\text{res}}^+ \\ &= \tau_{\text{res}}^- \cdot \tilde{L}^{-+} \cdot Z_l^{++} \cdot \tau_{\text{res}}^+, \end{aligned} \quad (119)$$

where renormalized vertices

$$\begin{aligned} \tau_{\text{res}}^{\pm} &= \tau_0^{\pm} \cdot [1 - \tilde{L}^{\pm\pm} \cdot \mathcal{G}_0^{\pm\pm}]^{-1} \\ &= [1 - \mathcal{G}_0^{\pm\pm} \cdot \tilde{L}^{\pm\pm}]^{-1} \cdot \tau_0^{\pm} \end{aligned} \quad (120)$$

are the solutions of Eq. (85) (and of similar equation for the $(+)$ vertex) with the omitted second term on the r.h.s. (within the RPA approximation). With the help of Eq. (119) expressed in terms of \tilde{L}^{ij} one can calculate the reaction rates associated with the processes described by Eq. (112) (in case of non-equilibrium slowly evolving systems with small spatial gradients).

2. Equilibrium systems

In equilibrium the expressions derived above can be simplified considerably and expressed through the real and imaginary parts of the function \tilde{L}^R :

$$\mathcal{G}_0 \cdot \tilde{L}^{-+} \cdot \mathcal{G}_0 = 2in_b(\omega) \Im(\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0), \quad (121a)$$

$$\mathcal{G}_0 \cdot \tilde{L}^{+-} \cdot \mathcal{G}_0 = 2i[1 + n_b(\omega)] \Im(\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0), \quad (121b)$$

$$\begin{aligned} \mathcal{G}_0 \cdot \tilde{L}^{--} \cdot \mathcal{G}_0 &= \mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0 \\ &+ 2in_b(\omega) \Im(\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0), \end{aligned} \quad (121c)$$

$$\begin{aligned} \mathcal{G}_0 \cdot \tilde{L}^{++} &= -\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0 \\ &+ 2i[n_b(\omega) + 1] \Im(\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0), \end{aligned} \quad (121d)$$

where $n_b(\omega) = [e^{\omega/T} - 1]^{-1}$ are the equilibrium boson occupations. These relations are derived in Appendix C.

We note that the self-energy (119) can be written as

$$\begin{aligned}\Sigma_{\text{RPA}}^{-+} = & \tau_0^- \cdot [\mathcal{G}_0^{--}]^{-1} \cdot \mathcal{G}_{\text{res}}^{--} \cdot \tilde{L}^{-+} \\ & \cdot [1 - \mathcal{G}_{\text{res}}^{++} \cdot \tilde{L}^{+-} \mathcal{G}_{\text{res}}^{--} \cdot \tilde{L}^{-+}]^{-1} \\ & \cdot [1 - \mathcal{G}_{\text{res}}^{++} \cdot \tilde{L}^{++}]^{-1} \cdot \tau_0^+, \quad (122)\end{aligned}$$

where with the help of the equilibrium relations (121) we express

$$\begin{aligned}\mathcal{G}_{\text{res}}^{--} \cdot \tilde{L}^{-+} \cdot \mathcal{G}_0 &= \frac{2i n_b}{1 + i(2n_b + 1)Y} \cdot Y \cdot \mathcal{G}_0, \\ \mathcal{G}_{\text{res}}^{++} \cdot \tilde{L}^{+-} \cdot \mathcal{G}_0 &= \frac{-2i(n_b + 1)}{1 - i(2n_b + 1)Y} \cdot Y \cdot \mathcal{G}_0, \\ [1 - \mathcal{G}_{\text{res}}^{++} \cdot \tilde{L}^{++}]^{-1} \cdot \mathcal{G}_0 &= [1 - i(2n_b + 1)Y]^{-1} \\ & \cdot [1 - \mathcal{G}_0^{--} \cdot \mathcal{R}\tilde{L}^R]^{-1} \cdot \mathcal{G}_0\end{aligned}\quad (123)$$

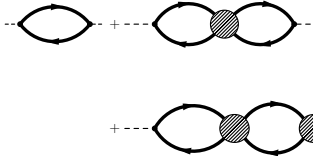
through the common matrix

$$Y = [1 - \mathcal{G}_0^{--} \cdot \mathcal{R}\tilde{L}^R]^{-1} \cdot \mathcal{G}_0^{--} \cdot \mathfrak{I}\tilde{L}^R.$$

Since functions of the matrix Y commute with each other, we can simplify Eq. (122) as

$$\begin{aligned}\Sigma_{\text{RPA}}^{-+} &= 2i n_b \tau_0^- \cdot \mathcal{G}_0^{-1} \cdot Y \cdot [1 + Y \cdot Y]^{-1} \\ & \cdot [1 - \mathcal{G}_0 \cdot \mathcal{R}\tilde{L}^R]^{-1} \cdot \tau_0^+ \\ &= 2i n_b \tau_0^- \cdot \mathcal{G}_0^{-1} \cdot [\mathcal{G}_0 \cdot \mathfrak{I}\tilde{L}^R + (1 - \mathcal{G}_0 \cdot \mathcal{R}\tilde{L}^R) \\ & \cdot [\mathcal{G}_0 \cdot \mathfrak{I}\tilde{L}^R]^{-1} \cdot (1 - \mathcal{G}_0 \cdot \mathcal{R}\tilde{L}^R)]^{-1} \cdot \tau_0^+. \quad (124)\end{aligned}$$

It is known that in the equilibrium the production rate can be also calculated with the help of the retarded self-energy. Note that in this case the expression (117) for Σ_{RPA}^R can be also obtained from the direct summation of the series of diagrams



$$(125)$$

where all quantities are defined for $T \neq 0$ within the standard Matsubara technique (using discrete frequencies $i\omega_n$). After the common replacement $i\omega_n \rightarrow \omega + i0$ one obtains the analytical continuation to the retarded function. Applying standard rules to take the real and imaginary parts of the inverse of a complex matrix⁵ from

$$\Re(M^{-1}) = -\frac{1}{\Im M} \Re M \Im(M^{-1}), \quad (126)$$

$$\Im(M^{-1}) = \frac{-1}{\Im M + \Re M \frac{1}{\Im M} \Re M}. \quad (127)$$

⁵ If for a complex matrix $M = \Re M + i\Im M$, there exist an inverse matrix M^{-1} then the real and imaginary parts of the inverse matrix are equal to

Eq. (117) we obtain

$$\begin{aligned}\Im \Sigma_{\text{RPA}}^R &= \tau_0 \cdot \mathcal{G}_0^{-1} \cdot \mathfrak{I}[1 - \mathcal{G}_0 \cdot \tilde{L}^R]^{-1} \cdot \tau_0 \\ &= \tau_0 \cdot \mathcal{G}_0^{-1} \cdot [\mathcal{G}_0 \cdot \mathfrak{I}\tilde{L}^R + (1 - \mathcal{G}_0 \mathcal{R}\tilde{L}^R) \\ & \cdot [\mathcal{G}_0 \mathfrak{I}\tilde{L}^R]^{-1} \cdot (1 - \mathcal{G}_0 \mathcal{R}\tilde{L}^R)]^{-1} \cdot \tau_0. \quad (128)\end{aligned}$$

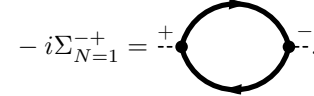
Comparing Eqs. (124) and (128) we recover the standard equilibrium relation

$$\Sigma^{-+} = -2i \Im \Sigma^R n_b, \quad (129)$$

see (C4). The sign minus compared to the corresponding relation for L , appears here since the self-energy includes vertices $\tau_0^- = -\tau_0^+ = \tau_0$.

Thus, we have shown that in case of equilibrium systems the reaction rates can be found either by using Matsubara technique and then recovering $\Im \Sigma_{\text{RPA}}^R$ given by diagrams (125), or equivalently by summing up series of diagrams (112) within the Schwinger-Keldysh formalism. *We emphasize that in the latter case not only the $N = 1$ term but also corresponding multi-piece diagrams must be included.*

Let us now analyze the contribution of genuine one-nucleon processes separated according to the $G^{\pm\mp}$ counting introduced in Sec. IV B. They are represented by the only diagram (80)



$$-i\Sigma_{N=1}^{-+} = \text{---}^+ \text{---}^- \quad (130)$$

It produces the expression

$$\begin{aligned}\Sigma_{N=1}^{-+} &= \tau_{\text{res}}^- \cdot \tilde{L}^{-+} \cdot \tau_{\text{res}}^+ = \tau_0^- \cdot [\mathcal{G}_0^{--}]^{-1} \cdot \mathcal{G}_{\text{res}}^{--} \\ & \cdot \tilde{L}^{-+} [1 - \mathcal{G}_{\text{res}}^{++} \cdot \tilde{L}^{++}]^{-1} \cdot \tau_0^+. \quad (131)\end{aligned}$$

Making use of Eq. (123) we obtain

$$\begin{aligned}\Sigma_{N=1}^{-+} &= 2i n_b \tau_0^- \cdot \mathcal{G}_0^{-1} \cdot [(2n_b + 1)^2 \mathcal{G}_0 \cdot \mathfrak{I}\tilde{L}^R \\ & + (1 - \mathcal{G}_0 \cdot \mathcal{R}\tilde{L}^R) \cdot [\mathcal{G}_0 \cdot \mathfrak{I}\tilde{L}^R]^{-1} \\ & \cdot (1 - \mathcal{G}_0 \cdot \mathcal{R}\tilde{L}^R)]^{-1} \cdot \tau_0^+. \quad (132)\end{aligned}$$

We see that in contrast to Eq. (124) there is an extra factor $(2n_b + 1)^2$ in the denominator of Eq. (132). Thus the relation (129) holds for $\Sigma_{N=1}^{-+}$ only approximately, e.g. for $n_b \ll 1$.

This example teaches us that multi-piece diagrams may yield a contribution to the total production rate, beyond that is given by the purely one-nucleon diagram (80).

Moreover, as we will show below, only for the rates calculated with Eqs. (112), (117) (and Eqs. (124), (128), respectively) the condition of the vector current conservation is exactly fulfilled. It would be fulfilled only approximately (e.g., for $n_b \ll 1$) or even violated in general case, if the rates were calculated according to Eq. (131). Bearing in mind that in many cases it is important to

keep conservation laws on exact level, provided it is possible, we may re-interpret which diagrams correspond to the one-nucleon, two-nucleon, and other processes: We will ascribe a diagram to the one-nucleon process, if after the cut through the $(\pm\mp)$ lines it decays into two pieces with two fermion legs, supplemented by the corresponding multi-piece terms. Diagrams producing two pieces with four fermion legs each plus the corresponding multi-piece terms, describe two-nucleon processes, etc. We stress that only taking multi-piece diagrams into account one recovers exact conservation laws (like the vector current conservation) in sub-sets of diagrams responsible to one-nucleon, two-nucleon processes, etc.

E. Extension to a superfluid system

In the system with pairing we have to deal with the larger number of interaction amplitudes and loop-functions \tilde{L} . There are altogether 16 quantities corresponding to the possible direction of the arrows. The uniform description can be achieved in the so-called "arrow space" introduced by A.J. Leggett in Ref. [16], where each element is labeled according to the direction of the arrows attached to it. We use label 1 for the arrow to the left ("+" in the Leggett's notations) and label 2 for the arrow to the right ("−" in the Leggett's notations). For instance the particle-hole interaction \mathcal{G}_0 and \mathcal{G} entering Eq. (103) will bring the indices $(\mathcal{G}_0)_{11}^{22}$. The 16 elements of the interaction amplitudes and the loop can be now arranged in the 4×4 matrix according to the basis

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}. \quad (133)$$

The bare interaction matrix combines the interactions in particle-hole $(\mathcal{G}_0)_{11}^{22}$, hole-particle $(\mathcal{G}_0)_{22}^{11}$, particle-particle $(\mathcal{G}_0)_{22}^{22}$, and hole-hole $(\mathcal{G}_0)_{11}^{11}$ channels arranged in the diagonal matrix

$$(\mathcal{G}_0)_{\mu\nu} = \begin{bmatrix} (\mathcal{G}_0)_{11}^{22} & & & \\ & (\mathcal{G}_0)_{22}^{11} & & \\ & & (\mathcal{G}_0)_{11}^{11} & \\ & & & (\mathcal{G}_0)_{22}^{22} \end{bmatrix}_{\mu\nu} = \begin{bmatrix} \text{diagram 1} & & & \\ & \text{diagram 2} & & \\ & & \text{diagram 3} & \\ & & & \text{diagram 4} \end{bmatrix}_{\mu\nu}. \quad (134)$$

The indices μ and ν run from 1 to 4. Note that interaction in particle-hole and particle-particle channels must

not be the same. The matrix of the \tilde{L} functions reads

$$\tilde{L}_{\mu\nu} = \begin{bmatrix} \tilde{L}_{22}^{11} & \tilde{L}_{21}^{12} & \tilde{L}_{21}^{11} & \tilde{L}_{22}^{12} \\ \tilde{L}_{12}^{21} & \tilde{L}_{11}^{22} & \tilde{L}_{11}^{21} & \tilde{L}_{12}^{22} \\ \tilde{L}_{12}^{11} & \tilde{L}_{11}^{12} & \tilde{L}_{11}^{11} & \tilde{L}_{12}^{12} \\ \tilde{L}_{22}^{21} & \tilde{L}_{21}^{22} & \tilde{L}_{21}^{21} & \tilde{L}_{22}^{22} \end{bmatrix}_{\mu\nu} = \begin{bmatrix} \text{diagram 1} & \text{diagram 2} & \text{diagram 3} & \text{diagram 4} \\ \text{diagram 5} & \text{diagram 6} & \text{diagram 7} & \text{diagram 8} \\ \text{diagram 9} & \text{diagram 10} & \text{diagram 11} & \text{diagram 12} \\ \text{diagram 13} & \text{diagram 14} & \text{diagram 15} & \text{diagram 16} \end{bmatrix}_{\mu\nu}. \quad (135)$$

In terms of the matrices \mathcal{G}_0 , \mathcal{G} and \tilde{L} Eq. (103) can be written as

$$\mathcal{G}_{\mu\nu}^{ji} = \mathcal{G}_{\mu\nu}^{ji} + \sum_{\gamma\delta} \sum_{kl} \mathcal{G}_{\mu\gamma}^{jl} \cdot \tilde{L}_{\gamma\delta}^{lk} \cdot (\mathcal{G}_0)_{\delta\nu}^{ki} \quad (136)$$

in double-index notations, where the Latin indices run in the Schwinger-Keldysh space, $j, i = +, -$ and the Greek indices run over the basis (133) in the arrow space, $\mu, \nu = 1, 2, 3, 4$. Since all above derivations were performed for the matrix object, the generalization of the results (107) and (111) is now given in terms of the nested matrices (134) and (135).

The equation for the self-energy (113) takes the following form

$$\Sigma^{ij} = \tau_0^{\dagger i} \cdot \tilde{L}^{ij} \cdot \tau_0^j + \sum_{kl} \tau_0^{\dagger i} \cdot \tilde{L}^{ik} \cdot \mathcal{G}^{kl} \cdot \tilde{L}^{lj} \cdot \tau_0^j, \quad (137)$$

where τ_0^j is the array in the arrow space $\tau_0^j = (\tau_0^{h,j}, \tau_0^j, 0, 0)$. The result (119) will be expressed now through the renormalized vertex $\tau_{\text{res}}^j = (\tau^{h,j}, \tau^j, \tau^{(1)j}, \tau^{(2)j})$, obeying the equation

$$\tau_{\text{res}}^{\pm} \cdot [1 - \mathcal{G}_0^{\pm\pm} \cdot \tilde{L}^{\pm\pm}]^{-1} = \tau_0^{\pm}, \quad (138)$$

which has the diagrammatic representation as in Eq. (46).

F. Application to the point-like interactions

To present the matrix relations derived above in a more transparent form, we consider a model case of a momentum independent bare interaction, \mathcal{G}_0^{ij} , and bare vertex, τ_0^j . The matrix Eqs. (107) turn into the algebraic ones with the replacement $\mathcal{G}_0 \cdot \tilde{L}^{ij} \cdot \mathcal{G}_0 \rightarrow \mathcal{G}_0^2 L^{ij}$. The expressions derived in this section become very compact in this case.

1. Non-equilibrium systems

The series of diagrams (125) for the Σ^R is easily summed up to the known result

$$\Sigma_{\text{RPA}}^R = \frac{\Sigma_0^R}{1 - \tilde{\mathcal{G}}_0 \Sigma_0^R}, \quad (139)$$

$$\Im \Sigma_{\text{RPA}}^R = \frac{\Im \Sigma_0^R}{(1 - \tilde{\mathcal{G}}_0 \Re \Sigma_0^R)^2 + (\tilde{\mathcal{G}}_0 \Im \Sigma_0^R)^2}. \quad (140)$$

In order to simplify subsequent diagrammatic representation we introduced the new object $\tilde{\mathcal{G}} = \mathcal{G}/\tau_0^2$.

On the other hand, from (107) we find

$$\tilde{\mathcal{G}}^{-+} = \frac{-\tilde{\mathcal{G}}_0^2 \Sigma_0^{-+}}{[1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}][1 + \tilde{\mathcal{G}}_0 \Sigma_0^{++}] + \tilde{\mathcal{G}}_0^2 \Sigma_0^{+-} \Sigma_0^{-+}}. \quad (141)$$

Similarly, for Σ_{RPA}^{--} diagrams in Eq. (112) can be re-grouped as

$$-i\Sigma_{\text{RPA}}^{--} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---}.$$

Thus

$$\begin{aligned} \Sigma_{\text{RPA}}^{--} &= \frac{\Sigma_0^{--}}{1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}} + \frac{\Sigma_0^{--}}{1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}} \tilde{\mathcal{G}}^{-+} \Sigma_0^{+-} + \frac{\Sigma_0^{--}}{1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}} \tilde{\mathcal{G}}^{++} \Sigma_0^{+-} \\ &= \frac{\Sigma_0^{--} + \tilde{\mathcal{G}}_0 (\Sigma_0^{--} \Sigma_0^{++} - \Sigma_0^{-+} \Sigma_0^{+-})}{[1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}][1 + \tilde{\mathcal{G}}_0 \Sigma_0^{++}] + \tilde{\mathcal{G}}_0^2 \Sigma_0^{+-} \Sigma_0^{-+}}. \end{aligned} \quad (145)$$

Obviously Σ_{RPA}^{++} is found from (145) after the simultaneous replacements $(--)\leftrightarrow(++)$ and $(-+)\leftrightarrow(+ -)$. Also Σ_{RPA}^{--} (145) and Σ_{RPA}^{+-} can be recovered from (143) and (139) with the help of the relations (A5).

2. Equilibrium systems

Let us now consider an equilibrium system. Using the equilibrium relations between Σ_0^{ij} and the retarded loop Σ_0^R self-energies:

$$\begin{aligned} \Sigma_0^{-+} &= 2i\Im \Sigma_0^R n_b, \quad \Sigma_0^{+-} = 2i\Im \Sigma_0^R (1 + n_b), \\ \Sigma_0^{--} &= \Re \Sigma_0^R + i\Im \Sigma_0^R (2n_b + 1), \\ \Sigma_0^{++} &= -\Re \Sigma_0^R + i\Im \Sigma_0^R (2n_b + 1) \end{aligned} \quad (146)$$

and comparing (140) and (144) we see that they satisfy exactly the standard equilibrium self-consistency relation (129). Adding (144) and (145) we recover (139).

Now consider $N = 1$ term

$$\Sigma_{N=1}^{-+} = \frac{\Sigma_0^{-+}}{[1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}][1 + \tilde{\mathcal{G}}_0 \Sigma_0^{++}]}, \quad (147)$$

It is convenient to re-group diagrams for Σ_{RPA}^{-+} in Eq. (112) as

$$-i\Sigma_{\text{RPA}}^{-+} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \quad (142)$$

Thus, Σ^{-+} satisfies the equation

$$\Sigma_{\text{RPA}}^{-+} = \frac{\Sigma_0^{-+}}{[1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}][1 - \tilde{\mathcal{G}}_0 \Sigma_0^{++}]} + \Sigma_0^{-+} \tilde{\mathcal{G}}^{+-} \Sigma_0^{-+}. \quad (143)$$

Using (107) we find

$$\Sigma_{\text{RPA}}^{-+} = \frac{\Sigma_0^{-+}}{[1 - \tilde{\mathcal{G}}_0 \Sigma_0^{--}][1 + \tilde{\mathcal{G}}_0 \Sigma_0^{++}] + \tilde{\mathcal{G}}_0^2 \Sigma_0^{+-} \Sigma_0^{-+}}. \quad (144)$$

where we took into account opposite signs of the bare interaction with the + and - signs. Using the equilibrium relations (146) we obtain

$$\begin{aligned} \Sigma_{N=1}^{-+} &= \frac{-2i\Im \Sigma_0^R n_b}{(1 - \tilde{\mathcal{G}}_0 \Re \Sigma_0^R)^2 + \tilde{\mathcal{G}}_0^2 (\Im \Sigma_0^R)^2 - (\tilde{\mathcal{G}}_0)^2 \Sigma_0^{-+} \Sigma_0^{+-}} \\ &= \frac{-2i\Im \Sigma_0^R n_b}{(1 - \tilde{\mathcal{G}}_0 \Re \Sigma_0^R)^2 + [(2n_b + 1)\tilde{\mathcal{G}}_0 \Im \Sigma_0^R]^2}. \end{aligned} \quad (148)$$

Replacing (140) and (148) in (129) we see that the latter condition may hold for $N = 1$ term only approximately, e.g. for $n_b \ll 1$. Also $\Sigma_{N=1}^{-+} \simeq \Sigma_{\text{RPA}}^{-+}$ provided $|(2n_b + 1)\tilde{\mathcal{G}}_0 \Im \Sigma_0^R| \ll |1 - \tilde{\mathcal{G}}_0 \Re \Sigma_0^R|$ that holds for $T \ll \epsilon_F$ in most interesting for us cases. However as we have stressed, it might be of principal importance to keep exact conservation laws, like the condition of the conservation of the vector current. As we show below, the latter condition is exactly satisfied provided one uses Σ_{RPA} rather than $\Sigma_{N=1}$.

V. RENORMALIZATION OF INTERACTION

The results of the previous section show that the interaction \mathcal{G}_0 enters the resulting expression for the resummed interaction \mathcal{G}^{ij} [Eq. (107)] and the self-energy Σ^{-+} [Eq. (119)] together with the \tilde{L}^{--} or \tilde{L}^{++} functions in specific combinations given by the residual interaction $\mathcal{G}_{\text{res}}^{\pm\pm}$ [Eq. (105)] and the residual vertex τ_{res}^{\pm} [Eq. (120)]. The structure of Eqs. (105,120) suggests that if one is able to split an *a priori* complicated function \tilde{L}^{ii} into some part, which we call “known”, $\tilde{L}_{\text{known}}^{ii}$, and a remainder $\tilde{L}_{\text{rem}}^{ii} = \tilde{L}^{ii} - \tilde{L}_{\text{known}}^{ii}$, then the quantities $\mathcal{G}_{\text{res}}^{ii}$ and τ_{res}^i can be expressed through the “known” part $\tilde{L}_{\text{known}}^{ii}$,

$$\begin{aligned}\mathcal{G}_{\text{res}}^{\pm\pm} &= [1 - \mathcal{G}_{\text{ren}}^{\pm\pm} \cdot \tilde{L}_{\text{known}}^{\pm\pm}]^{-1} \cdot \mathcal{G}_{\text{ren}}^{\pm\pm}, \\ \tau_{\text{res}}^{\pm} &= [1 - \mathcal{G}_{\text{ren}}^{\pm\pm} \cdot \tilde{L}_{\text{known}}^{\pm\pm}]^{-1} \cdot \tau_{\text{ren}}^{\pm},\end{aligned}\quad (149)$$

and the renormalized interaction and vertex

$$\begin{aligned}\mathcal{G}_{\text{ren}}^{\pm\pm} &= [1 - \mathcal{G}_0^{\pm\pm} \cdot \tilde{L}_{\text{rem}}^{\pm\pm}]^{-1} \cdot \mathcal{G}_0^{\pm\pm}, \\ \tau_{\text{ren}}^{\pm} &= [1 - \mathcal{G}_0^{\pm\pm} \cdot \tilde{L}_{\text{rem}}^{\pm\pm}]^{-1} \cdot \tau_0^{\pm}.\end{aligned}\quad (150)$$

By a cunning choice of the “known” part one can account in $\tilde{L}_{\text{known}}^{ii}$ for the most rapid variations with the energy and momentum in the interval of interest. Then the renormalized quantities (150) will possess a weak energy-momentum dependence and can be cast in terms of phenomenological parameters adjusted to some empirical data. For nuclear physics such a renormalization program was conducted by A.B. Migdal in his seminal paper [5].

A. Fermi-liquid renormalization

Simplifying consideration we focus below on the description of equilibrium systems. Then we may deal with only one, e.g. retarded Green’s function. The other Green’s functions G_n^{-} , G_n^{+} , G_n^{+-} , and G_n^{++} are expressed in equilibrium through the retarded Green’s function, see Appendix C.

At low temperatures of our interest ($T \lesssim \Delta \ll \varepsilon_{\text{F},n}, \varepsilon_{\text{F},p}$) neutrons and protons are only slightly excited above their Fermi seas and all the processes occur in a narrow vicinity of $\varepsilon_{\text{F},n}$ and $\varepsilon_{\text{F},p}$. In such a situation the Fermi-liquid approach seems to be the most efficient one. The basic assumption of the Fermi-liquid renormalization is that in a fermion system there is some mechanism of single-particle excitations. For normal systems at $T = 0$ this manifests itself in a jump in the particle momentum distribution⁶. According to the A.B. Migdal’s paper [3]

this jump indicates the presence of a pole in the fermion Green’s function. Thus, the full retarded Green’s function in the momentum representation is given by a sum of a pole term and a regular part G_{reg} ,

$$G_n^R(\epsilon, \vec{p}) = \frac{a}{\epsilon - \epsilon_p + i\gamma\epsilon^2} + G_{\text{reg}}^R(\epsilon, \vec{p}). \quad (151)$$

Here and below the energy $\epsilon = p_0 - \epsilon_{\text{F}}$ is counted from the Fermi energy and the kinetic energy is $\epsilon_p = \frac{p^2 - p_{\text{F},N}^2}{2m_N^*}$. The residue of the pole term, a , quantifies the A.B. Migdal’s jump in the momentum distribution

$$a^{-1} = 1 - \left(\frac{\partial \Sigma_n^R(\epsilon, \vec{p})}{\partial \epsilon} \right)_{0, p_{\text{F},N}}, \quad (152)$$

where one can put $T = 0$ which is correct up to higher order contribution $O(T^2/\epsilon_{\text{F},N}^2)$.

The in-medium mass of the fermion is given by

$$\frac{1}{m_N^*} = a \left(\frac{1}{m_N} + 2 \frac{\partial \Sigma_n^R(\epsilon, \vec{p})}{\partial p^2} \right)_{0, p_{\text{F},N}}, \quad (153)$$

and the pole width is because of the coupling to the two-particle-hole mode,

$$\gamma = -\Im \Sigma_n^R(\epsilon, p_{\text{F},N})/\epsilon^2 \simeq \text{const}, \quad (154)$$

for $\epsilon \ll \epsilon_{\text{F},N}$. At finite temperature we have to replace $\epsilon_{\text{F},N} \rightarrow \mu_N \simeq \epsilon_{\text{F},N} + O(T^2/\epsilon_{\text{F},N}^2)$ and $p_{\text{F},N} \rightarrow \sqrt{2m_N\mu_N}$ but these corrections are small for $T \lesssim \Delta$ and can be neglected.

The typical averaged outgoing neutrino energy ($\omega \sim T$) is larger than the nucleon particle width $\Gamma_N(\epsilon \sim T) \sim \gamma T^2 \sim T^2/\varepsilon_{\text{F},N}$. Therefore, one can neglect the width in the pole term in Eq. (151) and work within the quasi-particle Green’s function

$$G_{n,\text{q.p.}}^R(p) = G_{n,\text{q.p.}}^R(\epsilon, \vec{p}) = \frac{a}{\epsilon - \epsilon_p + i0}. \quad (155)$$

Following [6] only the pole part of G_n^R is relevant for descriptions of processes happening in a weakly excited Fermi system. The regular part can be absorbed by the renormalization of the particle-particle and particle-hole interactions on the Fermi surface.

In the momentum representation the particle-particle interaction (38) depends on spins and momenta of incoming ($a, q/2 - p$; $b, q/2 + p$) and outgoing ($c, q/2 - p'$; $d, q/2 + p'$) particles $[\hat{V}^{--}(\frac{q}{2} + p, \frac{q}{2} - p; \frac{q}{2} + p', \frac{q}{2} - p')]_{cd,ab} = [\hat{V}^{--}(p, p', q)]_{cd,ab}$, where we introduce the total momentum of two particles q and relative momenta in incoming and outgoing channels, p and p' . The particle-hole interaction (50) depends on spins and momenta of incoming ($b, p + q/2$; $a, p - q/2$) and outgoing ($d, p' + q/2$; $c, p' - q/2$) particles and holes, respectively, $[\hat{U}^{--}(p + \frac{q}{2}, p - \frac{q}{2}; p' + \frac{q}{2}, p' - \frac{q}{2})]_{dc,ab} =$

⁶ There is a special class of Fermi systems in which the jump is absent even in the normal state. They are called singular Fermi liquids or Non-Fermi liquids [127].

$[\hat{U}^{--}(p, p', q)]_{dc, ab}$. Graphical Eqs. (35), (49) can be written for causal functions:

$$\begin{aligned} \hat{T}_{pp}^{--}(p, p'; q) &= \hat{V}^{--}(p, p'; q) + \int \frac{d^4 p''}{(2\pi)^4 i} \\ &\times \hat{V}^{--}(p, p''; q) \hat{G}_n^{--}(p''_+) \hat{G}_n^{--}(-p''_-) \hat{T}_{pp}^{--}(p'', p'; q), \end{aligned} \quad (156)$$

$$\begin{aligned} \hat{T}_{ph}^{--}(p, p'; q) &= \hat{U}^{--}(p, p'; q) + \int \frac{d^4 p''}{(2\pi)^4 i} \\ &\times \hat{U}^{--}(p, p''; q) \hat{G}_n^{--}(p''_+) \hat{G}_n^{--}(p''_-) \hat{T}_{ph}^{--}(p'', p'; q), \end{aligned} \quad (157)$$

where $p''_{\pm} = (\epsilon \pm \omega/2, \vec{p}'' \pm \vec{q}/2)$. The integration over p'' involves energies far off the Fermi surface. One may renormalize interactions \hat{V} in (156) and \hat{U} in (157) such that integration will go over the region near the Fermi surface and one may use simple quasi-particle expressions for the Green's functions. Integral over internal momenta can be written now as

$$\int \frac{d^4 p}{(2\pi)^4 i} (\dots) \simeq \int d\Phi_0 \langle \dots \rangle_{\vec{n}},$$

where for $T = 0$ the energy momentum integral is as follows

$$\int d\Phi_0 = \rho \int_{-\infty}^{+\infty} \frac{d\epsilon}{2\pi i} \int_{-\infty}^{+\infty} d\epsilon_p, \quad (158)$$

and the brackets stand for averaging over the momentum direction $\vec{n} = \vec{p}/|\vec{p}|$

$$\langle \dots \rangle_{\vec{n}} = \int \frac{d\Omega_{\vec{n}}}{4\pi} (\dots) \quad (159)$$

with $\rho = \frac{m_N^* p_{F,N}}{\pi^2}$ being the density of states at the Fermi surface. If the interaction \hat{V} is not singular at $q \rightarrow 0$, then for small q and $|\vec{p}| \simeq p_{F,N} \simeq |\vec{p}'|$ the amplitude \hat{T}_{pp} is a function of $\cos \theta_{pp'} = (\vec{n}\vec{n}')$, $\vec{n}' = \vec{p}'/|\vec{p}'|$, and Eq. (156) can be written as

$$\begin{aligned} \hat{T}_{pp}^{--}(\vec{n}', \vec{n}) &= \hat{\Gamma}^{\xi}(\vec{n}', \vec{n}) \\ &+ \mathcal{A}_{pp} \langle \hat{\Gamma}^{\xi}(\vec{n}', \vec{n}'') \hat{T}_{pp}^{--}(\vec{n}'', \vec{n}) \rangle_{\vec{n}''}. \end{aligned} \quad (160)$$

The information about the bare interaction and the structure of Green's functions in regions far from the Fermi surface is encoded in the effective interactions

$$\begin{aligned} \hat{\Gamma}^{\xi}(\vec{n}', \vec{n}) &= \hat{V}^{--}(\vec{p}'_F, \vec{p}_F) + \int \frac{d^4 p''}{(2\pi)^4 i} \theta(\epsilon_{p''} - \xi) \\ &\times \hat{V}^{--}(\vec{p}'_F, \vec{p}''_F) \hat{G}_n^{--}(p'') \hat{G}_n^{--}(-p'') \hat{\Gamma}^{\xi}(\vec{n}'', \vec{n}). \end{aligned}$$

The quantity ξ ($\Delta \ll \xi \ll \epsilon_F$) does not appear in final expressions, except for the equation expressing the gap in terms of the phenomenological Landau-Migdal parameter, see Eq. (188) below. The \mathcal{A}_{pp} function in the quasi-particle approximation for the Green's function (151),

$$G_{n,q,p}^{--}(p) = \frac{a}{\epsilon - \epsilon_p + i0 \text{sign} \epsilon}, \quad (161)$$

renders

$$\mathcal{A}_{pp} = \int d\Phi_0 G_{n,q,p}^{--}(+p) G_{n,q,p}^{--}(-p) \theta(\xi - \epsilon_p).$$

The similar renormalization can be performed with Eq. (157). Here we have to note that even if \hat{U} changes weakly for small q , the particle-hole scattering amplitude is a sharp function of q because poles in $G_n^{--}(p_+) G_n^{--}(p_-)$ approach each other producing $\delta(\epsilon)$:

$$\begin{aligned} G_n^{--}(p_+) G_n^{--}(p_-) \\ \simeq \delta(\epsilon) \int d\epsilon G_{n,q,p}^{--}(p_+) G_{n,q,p}^{--}(p_-) + \mathcal{B}(p, q), \end{aligned}$$

and $\mathcal{B}(p, q)$ is a smooth function of p and q . Then, Eq. (157) can be rewritten near the Fermi surface, $|\vec{p}| \simeq p_{F,N} \simeq |\vec{p}'|$, as

$$\begin{aligned} \hat{T}_{ph}^{--}(\vec{n}', \vec{n}; q) &= \hat{\Gamma}^{\omega}(\vec{n}', \vec{n}) \\ &+ \langle \hat{\Gamma}^{\omega}(\vec{n}', \vec{n}'') \mathcal{A}_{ph}^{--}(\vec{n}'', q) \hat{T}_{ph}^{--}(\vec{n}'', \vec{n}; q) \rangle_{\vec{n}''}. \end{aligned} \quad (162)$$

The renormalized interaction is determined by the equation

$$\begin{aligned} \hat{\Gamma}^{\omega}(\vec{n}', \vec{n}) &= \hat{U}^{--}(\vec{p}'_F, \vec{p}_F) + \int \frac{d^4 p''}{(2\pi)^4 i} \\ &\times \hat{U}^{--}(\vec{p}'_F, \vec{p}''_F) \mathcal{B}(p_F, q=0) \hat{\Gamma}^{\omega}(\vec{n}'', \vec{n}), \end{aligned}$$

where we assume $|\vec{q}\vec{p}|/m_N^* \ll \omega \ll \mu$ and

$$\begin{aligned} \mathcal{B}(p_F, q=0) &= \lim_{\omega \rightarrow 0} \lim_{\vec{q} \rightarrow 0} [G_n^{--}(p_{F+}) G_n^{--}(p_{F-})] |, \\ p_{F\pm} &= (\pm\omega/2, p_F \vec{n} \pm \vec{q}/2). \end{aligned}$$

The particle-hole loop function in Eq. (162) is defined as

$$\mathcal{A}_{ph}^{--}(\vec{n}; q) = \int d\Phi_0 \hat{G}_{n,q,p}^{--}(p''_F) \hat{G}_{n,q,p}^{--}(p''_F). \quad (163)$$

We emphasize that the amplitudes $\hat{\Gamma}^{\xi}$ and $\hat{\Gamma}^{\omega}$ are not calculable within a Fermi liquid approach and should be considered as phenomenological quantities.

The renormalized amplitudes $\hat{\Gamma}^{\xi, \omega}$ can serve as a bare interaction

$$\mathcal{G}_0^{--} = \hat{\Gamma}^{\xi, \omega}, \quad \mathcal{G}_0^{++} = -\hat{\Gamma}^{\xi, \omega}$$

in the analysis performed in the previous section.

The renormalization procedure outlined in this section can be applied to a superfluid system without any modification, if the system is only slightly excited, $T \lesssim \Delta \ll \epsilon_F$. The regular quantities are subtracted at $T = 0$. The assumption of thermal equilibrium, which we used, allows to deal with the only one Green's function, here with G^{--} . If we deal with slightly non-equilibrium system for excitation energies $\epsilon^* \lesssim \Delta \ll \epsilon_F$, the same renormalization procedure should be performed for G^{-+} . Actually, it is more convenient to chose the spectral function

$A = -\Im G^R$ as another independent quantity, rather than G^{-+} , and perform renormalization of A ($G^{-+} = Af$, where f is the distribution function satisfying the generalized kinetic equation, see Ref. [107]). After the renormalization one may deal only with the quasiparticle Green's functions and with the renormalized interaction.

B. Landau-Migdal parameters for the nuclear matter

Because of the diagonal spin structure of the normal Green's functions, the spin structure of the amplitudes reflects the structure of bare interactions \hat{V} and \hat{U} :

$$[\hat{\Gamma}^\xi(\vec{n}', \vec{n})]_{cd,ab} = \Gamma_0^\xi(\vec{n}', \vec{n}) (i\sigma_2)_{dc} (i\sigma_2)_{ab} + \Gamma_1^{\xi,\alpha\beta}(\vec{n}', \vec{n}) (\vec{\sigma}^\alpha i\sigma_2)_{dc} (i\sigma_2 \vec{\sigma}^\beta)_{ab}, \quad (164)$$

$$[\hat{\Gamma}^\omega(\vec{n}', \vec{n})]_{dc,ab} = \Gamma_0^\omega(\vec{n}', \vec{n}) (\sigma_0)_{dc} (\sigma_0)_{ab} + \Gamma_1^{\omega,\alpha\beta}(\vec{n}', \vec{n}) (\vec{\sigma}^\beta)_{dc} (\vec{\sigma}^\alpha)_{ab}. \quad (165)$$

In the graphical form the quantity Γ^ω corresponds to the empty block in Eq. (10). In the absence of the spin-orbit coupling the interaction is invariant under independent rotation in spin and orbital spaces. In this case $\Gamma_1^{\xi(\omega),\alpha\beta} = \Gamma_1^{\xi(\omega)} \delta^{\alpha\beta}$. Oppositely, in a isotropic system the interaction containing the spin-orbit coupling is invariant under the combined rotations in both spaces and the spin structure of the interaction is more involved. According to Ref. [128] the general structure of the interaction in the system with one type of fermions can be written as

$$\begin{aligned} \hat{\Gamma}^\omega(\vec{n}', \vec{n}) &= \Gamma_0^\omega \sigma'_0 \sigma_0 + \Gamma_1^\omega (\vec{\sigma}' \vec{\sigma}) \\ &+ \Gamma_T^\omega [3 (\vec{\sigma}' (\vec{n} - \vec{n}')) (\vec{\sigma} (\vec{n} - \vec{n}')) - (\vec{\sigma}' \vec{\sigma})] \\ &\quad \times (\vec{n} - \vec{n}')^2 \\ &+ \Gamma_+^\omega [(\vec{\sigma}' \vec{n}) (\vec{\sigma} \vec{n}') + (\vec{\sigma}' \vec{n}') (\vec{\sigma} \vec{n})] \\ &+ \Gamma_-^\omega [(\vec{\sigma}' \vec{n}) (\vec{\sigma} \vec{n}') - (\vec{\sigma}' \vec{n}') (\vec{\sigma} \vec{n})] \\ &+ \Gamma_K^\omega (\vec{\sigma}' [\vec{n} \times \vec{n}']) (\vec{\sigma} [\vec{n} \times \vec{n}']). \end{aligned} \quad (166)$$

Each pre-factor here is a function of (\vec{n}', \vec{n}) . The spin index assignment is the same as in Eq. (165). Matrices $\sigma_j = (\sigma_j)_{ab}$ with $j = 0, \dots, 3$, act on the incoming fermions, while matrices $\sigma'_j = (\sigma'_j)_{dc}$ act on the outgoing fermions. The similar decomposition can be written also for $\hat{\Gamma}^\xi(\vec{n}', \vec{n})$ with the only replacements $\sigma_j \rightarrow i\sigma_2 \sigma_j$ and $\sigma'_j \rightarrow \sigma'_j i\sigma'_2$.

The renormalized amplitudes, $\Gamma_a^{\omega,\xi}$ in Eq. (166), where $a = 0, 1, \pm, T, K$, can be expanded in Legendre polynomials

$$\Gamma_a^{\omega(\xi)}(\vec{n}', \vec{n}) = \sum_l \Gamma_{a;l}^{\omega(\xi)} P_l(\vec{n}' \vec{n}). \quad (167)$$

The structure with Γ_T^ω is the tensor interaction considered in Ref. [129–131]. The effect of the tensor terms on

some of the static properties of nuclear and neutron matter was found to be very small [129]. However they could play an important role in the condition for stability of the ground state of nuclear matter [130]. The spin-orbit and tensor terms in Γ^ξ are important for the description of $3P_2$ pairing [79]. A non-local contribution of the pion exchange to Γ^ξ and its effect on nucleon superfluidity were studied in Ref. [132]. For the description of the majority of nuclear phenomena including the $1S_0$ pairing it is sufficient to consider only the terms with $\Gamma_0^{\omega(\xi)}$ and Γ_1^ω . In further we consider only these terms.

Following Ref. [6] we introduce the dimensionless parameters

$$\bar{f}_l^\omega = \frac{\Gamma_{0;l}^\omega}{a^2(n_0) \rho(n_0)}, \quad \bar{g}_l^\omega = \frac{\Gamma_{1;l}^\omega}{a^2(n_0) \rho(n_0)}, \quad \bar{f}_l^\xi = \frac{\Gamma_{0;l}^\xi}{a^2(n_0) \rho(n_0)}, \quad \bar{g}_l^\xi = \frac{\Gamma_{1;l}^\xi}{a^2(n_0) \rho(n_0)}, \quad (168)$$

where \bar{f} 's and \bar{g} 's are constants. We considered the system with the one type of particles, e.g. neutron matter. In general case of the nuclear matter of an arbitrary isotopic composition, we need to know the nn , pp and np interaction amplitudes. These quantities can be parameterized as

$$\begin{aligned} a^2(n_0) \rho(n_0) \hat{\Gamma}_{N_1 N_2}^\omega &= \bar{f}_{N_1 N_2}^\omega \sigma'_0 \sigma_0 + \bar{g}_{N_1 N_2}^\omega (\vec{\sigma}' \vec{\sigma}), \\ a^2(n_0) \rho(n_0) \hat{\Gamma}_{N_1 N_2}^\xi &= \bar{f}_{N_1 N_2}^\xi (i\sigma'_2) (i\sigma_2) \\ &+ \bar{g}_{N_1 N_2}^\xi (\vec{\sigma}' i\sigma'_2) (i\sigma_2 \vec{\sigma}), \end{aligned} \quad (169)$$

where $N_1, N_2 = n, p$. The calculation of the Landau-Migdal parameters is a formidable task and the results vary essentially, depending on a calculation scheme and a model for the bare nucleon-nucleon interaction, see, e.g., Refs. [76, 133–136]. Another possible path is to try to extract the Landau-Migdal parameters from the analysis of phenomena in atomic nuclei. Starting from Ref. [6] one traditionally presents the nucleon-nucleon interaction amplitudes in the form, cf. [137],

$$\begin{aligned} \hat{\Gamma}^\omega &= \frac{C}{a^2(n_0)} [f^\omega \sigma'_0 \sigma_0 + g^\omega (\vec{\sigma}' \vec{\sigma}) \\ &+ f'^\omega \vec{\tau}' \vec{\tau} \sigma'_0 \sigma_0 + g'^\omega (\vec{\tau}' \vec{\tau}) (\vec{\sigma}' \vec{\sigma})], \end{aligned} \quad (170)$$

$$\begin{aligned} \hat{\Gamma}^\xi &= \frac{C}{a^2(n_0)} [f^\xi \sigma'_0 \sigma_0 + g^\xi (\vec{\sigma}' \vec{\sigma}) \\ &+ f'^\xi (\vec{\tau}' \vec{\tau}) \sigma'_0 \sigma_0 + g'^\xi (\vec{\tau}' \vec{\tau}) (\vec{\sigma}' \vec{\sigma})]. \end{aligned} \quad (171)$$

The constant $C = 1/\rho(n_0)$ is introduced as a dimensional normalization factor. One usually fixes its value as $a^2(n_0) C = 300 \text{ MeV} \cdot \text{fm}^3$, see [28]. Making use of the parameterizations (170) and (171) one implicitly assumes that the Fermi-liquid renormalization preserves isospin symmetry of the strong interaction. Then, instead of six independent amplitudes $f_{N_1 N_2}^\omega$ and $g_{N_1 N_2}^\omega$ for nn , np and pp channels one deals with four amplitudes f^ω, g^ω and f'^ω, g'^ω . The others follow from the relations

$$\begin{aligned} \bar{f}_{nn}^\omega &= \bar{f}_{pp}^\omega = f^\omega + f'^\omega, \quad \bar{f}_{np}^\omega = \bar{f}_{pn}^\omega = f^\omega - f'^\omega, \\ \bar{g}_{nn}^\omega &= \bar{g}_{pp}^\omega = g^\omega + g'^\omega, \quad \bar{g}_{np}^\omega = \bar{g}_{pn}^\omega = g^\omega - g'^\omega. \end{aligned} \quad (172)$$

This assumption can be justified only for the nucleon matter with a *small isospin asymmetry*. For strongly asymmetrical nuclear matter, like the neutron star matter, the application of the relations (172) is questionable and should not hold *a priori*.

We note that for the particle-particle channel we use the spin parameterization (169) different from that in Eq. (171). The two sets of parameters are related as

$$\begin{aligned}\bar{f}_{nn}^\xi &= (f^\xi + f'^\xi) - 3(g^\xi + g'^\xi), \\ \bar{f}_{np}^\xi &= (f^\xi - f'^\xi) - 3(g^\xi - g'^\xi), \\ -\bar{g}_{nn}^\xi &= (f^\xi + f'^\xi) + (g^\xi + g'^\xi), \\ -\bar{g}_{np}^\xi &= (f^\xi - f'^\xi) + (g^\xi - g'^\xi).\end{aligned}\quad (173)$$

The values of the zero-th and first Legendre harmonics of f , f' , g , g' are extracted from analysis of many data on atomic nuclei. Unfortunately, there are essential uncertainties in numerical values of some of these parameters. These uncertainties are, mainly, due to attempts to get the best fit to experimental data in each specific case, modifying parameterization of the residual part of the NN interaction. Numerical values of the parameters extracted in Ref. [6] are $f_0^\omega \simeq 0.25$, $f_0'^\omega \simeq 1$, $g_0^\omega \simeq 0.5$, $g_0'^\omega \simeq 1$. Calculations in Ref. [138] give the values $f_0^\omega \simeq 0$, $f_0'^\omega \simeq 0.5-0.6$, $g_0^\omega \simeq 0.05 \pm 0.1$, $g_0'^\omega \simeq 1.1 \pm 0.1$. In Ref. [139] the value $g_{pp;0}$ was fixed by the data on the two-neutrino double β decays and the single β decays, as $g_{pp;0}^\omega \simeq 1$. This is in favor of the choice of Ref. [138]. First harmonics f_1^ω , $f_1'^\omega$ are related to the value of the effective nucleon mass. The values $g_{pp;1}^\omega = -g_{pn;1}^\omega \simeq -0.11$ are estimated from analysis of the decay energies and the Gamov-Teller strength distributions in neutron-rich short-lived nuclides [140]. In Ref. [141] the values $f_0^\xi \simeq -(0.25-0.33)$ and $g_0^\xi \simeq -(0.3-0.5)$ are extracted, whereas Ref. [97] calculated $f_0^\xi \simeq -0.47$ and $g_0^\xi \simeq +0.46$ using Cogy DSI force.

Pairing gaps depend on the density and are very sensitive to values of parameters in particle-particle channel because of exponential dependence on the interaction amplitude, see Refs. [142] and [84, 85]. In application to the pairing in neutron stars it seems to be preferable to use the values of the Landau-Migdal parameters adjusted to reproduce the pairing gaps obtained in microscopic calculations, like in Refs. [76-87, 97].

VI. EQUILIBRIUM SYSTEMS WITH PAIRING AT $T \neq 0$

In Section IV we have demonstrated that set of the diagrams for the one-nucleon process rate built up with the non-equilibrium Green's functions can be rewritten as the RPA series of the retarded self-energies. For the latter we may exploit the standard Fermi liquid approach. Thus further we follow the lines of [95].

A. Green functions and response of a system with pairing

As argued in the previous section, for $T \ll \epsilon_F$ we can use the quasiparticle approximation for the normal Green's function, once studied process occurs near the Fermi surface.

We assume that the renormalization procedure is properly done. Thus, we may deal only with the pole parts of the Green's functions characterized by the effective mass m^* and the residue a .

Then, neglecting $(T/\epsilon_F)^2$ corrections, for the retarded Green's function we write

$$\hat{G}_n^R(p) = G_n^R(p) \sigma_0, \quad G_n^R(p) = \frac{a}{\epsilon - \epsilon_p + i0}. \quad (174)$$

The Green's function of the hole is then given by

$$\hat{G}_n^{h,R}(p) = G_n^{h,R}(p) \sigma_0, \quad G_n^{h,R}(p) = G^A(-p). \quad (175)$$

Recall, here $p = (\epsilon, \vec{p})$. We will use the approximation $\epsilon_{\vec{p}+\vec{q}/2} \approx \epsilon_p + \vec{v} \vec{q}/2$, where \vec{v} is the nucleon velocity at the Fermi surface, $\vec{v} = v_F \vec{n}(1 + O(T^2/\epsilon_F^2))$. Actually, the denominators of the Green's functions are $|\omega \pm (\epsilon_{\vec{p}+\vec{q}/2} - \epsilon_{\vec{p}-\vec{q}/2})| \ll \epsilon_F$. Moreover, the terms $\propto \vec{v} \vec{q}$ may vanish under the angular integrations. Taking this into account we estimate that the neglected terms are at most of the order of $(\Delta/\epsilon_F)^2 \ll 1$ compared to the remained terms. Such corrections are usually omitted in most calculations within the Fermi-liquid theory for superfluids.

We need the gap function nearby the Fermi surface, hence, it is a function of the direction, \vec{n} , of the relative momentum of paired fermions

$$\begin{aligned}\hat{\Delta}^{(1)}(\vec{n}) &= (\Delta_0^{(1)}(\vec{n}) \sigma_0 + \vec{\Delta}_1^{(1)}(\vec{n}) \vec{\sigma}) i\sigma_2, \\ \hat{\Delta}^{(2)}(\vec{n}) &= i\sigma_2 (\Delta_0^{(2)}(\vec{n}) \sigma_0 + \vec{\Delta}_1^{(2)}(\vec{n}) \vec{\sigma}).\end{aligned}$$

As we argued in Section III the spin structure of the anomalous Green's functions repeats the spin structure of the gap functions

$$\begin{aligned}\hat{F}^{(1)}(p) &= (F_0^{(1)}(p) \sigma_0 + \vec{F}_1^{(1)}(p) \vec{\sigma}) i\sigma_2, \\ \hat{F}^{(2)}(p) &= i\sigma_2 (F_0^{(2)}(p) \sigma_0 + \vec{F}_1^{(2)}(p) \vec{\sigma}).\end{aligned}\quad (176)$$

Since, as we will see the gap is a sharp function of the temperature, we should retain this temperature dependence omitting it in other quantities.

In momentum representation Gor'kov's equations (31) for the retarded quantities render

$$\begin{aligned}\hat{G}_n^R(p) &= \hat{G}_n^R(p) + \hat{G}_n^R(p) \hat{\Delta}^{(1)R}(p, T) \hat{F}^{(2)R}(p), \\ \hat{F}^{(2)R}(p) &= \hat{G}_n^{hR}(p) \hat{\Delta}^{(2)R}(p, T) \hat{G}_n^R(p).\end{aligned}\quad (177)$$

Since $\hat{G}_n \propto \sigma_0$ one easily finds the solution

$$\begin{aligned}\hat{G}_n^R(p) &= \frac{\sigma_0 [G_n^{hR}(p)]^{-1}}{[G_n^R(p) G_n^{hR}(p)]^{-1} + \Delta^2(p, T)/a^2}, \\ \hat{F}^{(1,2)R}(p) &= \hat{\Delta}^{(1,2)R}(p, T) G_n^{hR}(p) G_n^R(p) \\ &= \frac{\hat{\Delta}^{(1,2)R}(p, T)}{[G_n^R(p) G_n^{hR}(p)]^{-1} + \Delta^2(p, T)/a^2}.\end{aligned}$$

We denote here

$$\begin{aligned}\Delta^2(p, T) &= -a^2 \frac{1}{2} \text{Tr} \{ \hat{\Delta}^{(2)R}(p, T) \hat{\Delta}^{(1)R}(p, T) \} \\ &= a^2 [\Delta_0^2(p, T) + \bar{\Delta}_1^2(p, T)].\end{aligned}\quad (178)$$

Then using that

$$[G_n(p) G_n^h(p)]^{-1} = [-(\epsilon + i0)^2 + \epsilon_p^2]/a^2, \quad (179)$$

we arrive at explicit expressions for the quasi-particle retarded Green's functions in the presence of pairing

$$\begin{aligned}G^R(p) &= a \frac{\epsilon + \epsilon_p}{(\epsilon + i0)^2 - E_p^2} \\ &= \frac{a u_p^2}{(\epsilon - E_p + i0)} + \frac{a v_p^2}{(\epsilon + E_p + i0)}, \\ \hat{F}^{(1)R}(p) &= F^R(p) \frac{\hat{\Delta}^{(1)}}{\Delta}, \quad \hat{F}^{(2)}(p) = F^R(p) \frac{\hat{\Delta}^{(2)}}{\Delta}, \\ F^R(p) &= \frac{-a \Delta(p, T)}{(\epsilon + i0)^2 - E_p^2} \\ &= \frac{-a u_p v_p}{(\epsilon - E_p + i0)} + \frac{a u_p v_p}{(\epsilon + E_p + i0)},\end{aligned}$$

where the Bogolyubov's factors are

$$u_p^2 = \frac{E_p + \epsilon_p}{2E_p}, \quad v_p^2 = \frac{E_p - \epsilon_p}{2E_p}, \quad (180)$$

and the quasi-particle excitations possess the gapped spectrum

$$E_p^2 = \epsilon_p^2 + \Delta^2(p, T). \quad (181)$$

After the retarded Green's functions are known other Green's functions can be expressed through them, see Appendix C. In the rest of the paper we consider the singlet $1S_0$ pairing, hence $\bar{\Delta}_1^{(1,2)} = 0$ and we will denote $\Delta = \Delta_0^{(1)} = \Delta_0^{(2)}$. So the causal Green's function at $T = 0$ reads

$$\begin{aligned}G^{--}(p) &= \frac{a(\epsilon + \epsilon_p)}{\epsilon^2 - E_p^2 + i0 \text{sgn} \epsilon}, \\ F^{--}(p) &= \frac{-a \Delta}{\epsilon^2 - E_p^2 + i0 \text{sgn} \epsilon}.\end{aligned}\quad (182)$$

In further we will calculate the production rate using the equilibrium relation (129). Hence we have to calculate the retarded self-energy at finite temperature. We can use the Matsubara technique with

$$G(p) = \frac{a(\epsilon_n + \epsilon_p)}{\epsilon_n^2 - E_p^2}, \quad F(p) = \frac{-a \Delta}{\epsilon_n^2 - E_p^2}, \quad (183)$$

where $\epsilon_n = i\pi(2n + 1)$, n is the integer number running from $-\infty$ to ∞ .

The energy-momentum integration at arbitrary temperature is defined as

$$\begin{aligned}\int d\Phi_T f(\epsilon, \epsilon_p) \\ = \begin{cases} \rho \int_{-\infty}^{+\infty} \frac{d\epsilon}{2\pi i} \int_{-\infty}^{+\infty} d\epsilon_p f(\epsilon, \epsilon_p) & \text{for } T = 0 \\ \rho T \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} d\epsilon_p f(i\epsilon_n, \epsilon_p) & \text{for } T \neq 0 \end{cases}.\end{aligned}\quad (184)$$

The singlet-pairing gap is determined by the Γ_0^ξ term in the particle-particle interaction, and the gap equation reads

$$\Delta(\vec{n}) = -\langle \Gamma_0^\xi(\vec{n}, \vec{n}') A_0(\Delta(\vec{n}')) \Delta(\vec{n}') \rangle_{\vec{n}'}, \quad (185)$$

$$A_0(\Delta) = \int d\Phi_T G_0(p) G^h(p) \theta(\xi - \epsilon_p), \quad (186)$$

where $G_0(p) = 1/(\epsilon_n - \epsilon_p)$ is the Matsubara Green's function for the Fermi system without pairing ($\Delta = 0$).

Note that the same value A_0 can be introduced as $A_0 = \int d\Phi_T (G(p) G^h(p) + F(p) F(p)) \theta(\xi - \epsilon_p)$, cf. Ref. [17].⁷ For vanishing temperature the direct calculation gives

$$\begin{aligned}\frac{A_0(\Delta)}{a^2 \rho} &= \int_{-\xi}^{+\xi} d\epsilon_p \int_{-\infty}^{+\infty} \frac{d\epsilon}{2\pi i} \frac{1}{\epsilon - \epsilon_p + i0} \frac{-\epsilon + \epsilon_p}{\epsilon^2 - E_p^2 + i0} \\ &= \int_{-\xi}^{+\xi} \frac{d\epsilon_p}{2\sqrt{\epsilon_p^2 + \Delta^2}} = \ln \left[\frac{\xi}{\Delta} + \sqrt{\frac{\xi^2}{\Delta^2} + 1} \right] \\ &\approx \ln \left[\frac{2\xi}{\Delta} \right], \quad \xi \gg \Delta.\end{aligned}\quad (187)$$

In case of the attractive interaction $f_0^\xi < 0$ the fermions are able to pair in $1S_0$ state and the relation between Δ , ξ and f_0^ξ is as follows,

$$\Delta = \xi \exp \left[-\frac{\rho(n_0)}{\rho(n) f_0^\xi} \right]. \quad (188)$$

In the renormalization procedure we used that $\xi \ll \epsilon_F$. However, fixing the gap, one usually puts $\xi = \epsilon_F$ bearing in mind the weak logarithmic dependence. For $T \neq 0$,

$$A_0(\Delta, T) = a^2 \rho \int_{-\xi}^{+\xi} \frac{d\epsilon_p (1 - 2n_f)}{2\sqrt{\epsilon_p^2 + \Delta^2}}, \quad (189)$$

where n_f is the Fermi distribution.

The diagrams for the self-energy are shown in Eq. (57). In momentum representation they produce the following

⁷ Definition of the value A_0 is here the same as in Ref. [15] and differs in sign from that used in Ref. [16, 17].

equation:

$$\begin{aligned}\Sigma &= \Sigma_0 + \Sigma_1, \\ \Sigma_0 &= \left\langle \int d\Phi_T t_0^\omega(\vec{n}, q) \left(G_+ G_- t_0(\vec{n}, q) \right. \right. \\ &\quad \left. \left. - F_+ F_- t_0^h(\vec{n}, q) + (G_+ F_- - F_+ G_-) \tilde{t}_0(\vec{n}, q) \right) \right\rangle_{\vec{n}}, \\ \Sigma_1 &= \left\langle \int d\Phi_T \tilde{t}_1^\omega(\vec{n}, q) \left(G_+ G_- \tilde{t}_1(\vec{n}, q) \right. \right. \\ &\quad \left. \left. + F_+ F_- \tilde{t}_1^h(\vec{n}, q) + (G_+ F_- - F_+ G_-) \tilde{\tilde{t}}_1(\vec{n}, q) \right) \right\rangle_{\vec{n}}.\end{aligned}$$

The left vertices here are the bare vertices (45) after the Fermi-liquid renormalization, like in Eq. (150). Their spin decomposition is $\hat{\tau}^\omega(\vec{n}, q) = t_0(\vec{n}, q)\sigma_0 + \tilde{t}_1(\vec{n}, q)\vec{\sigma}$. Doing calculations in Matsubara technique, we use notations: $G_+ = G(p_0 + \omega, \vec{p} + \vec{q}/2)$ and $G_- = G(p_0, \vec{p} - \vec{q}/2)$, and similarly for F_\pm functions. For continues frequencies we use the symmetrical 4-vector notations, i.e. $G_\pm = G(p \pm q/2)$ and, analogously, $F_\pm = F(p \pm q/2)$.

The full vertices, $t_0(\vec{n}, q)$ and $\tilde{t}_1(\vec{n}, q)$, are determined by the diagrammatic Eqs. (46). In Fig. 4 two of these equations are written explicitly in the momentum representation. These graphical equations were first introduced by A.I. Larkin and A.B. Migdal in Ref. [15].

B. Larkin-Migdal equations

The right vertices in Eq. (57) are the full in-medium-dressed vertices, which are functions of the out-going frequency ω and momentum \vec{q} , and the nucleon velocity $\vec{v} \simeq v_F \vec{n}$, $\vec{n} = \vec{p}/p$. Their spin structure is

$$\begin{aligned}\hat{\tau}(\vec{n}, q) &= t_0(\vec{n}, q)\sigma_0 + \vec{\sigma} \tilde{t}_1(\vec{n}, q), \\ \hat{\tau}^h(\vec{n}, q) &= t_0(-\vec{n}, q)\sigma_0 + \vec{\sigma}^T \tilde{t}(-\vec{n}, q), \\ \hat{\tau}^{(1)}(\vec{n}, q) &= (t_0^{(1)}(\vec{n}, q)\sigma_0 + \vec{\sigma} \tilde{t}_1^{(1)}(\vec{n}, q))i\sigma_2, \\ \hat{\tau}^{(2)}(\vec{n}, q) &= i\sigma_2 (t_0^{(2)}(\vec{n}, q)\sigma_0 + \vec{\sigma} \tilde{t}_1^{(2)}(\vec{n}, q)).\end{aligned}\tag{190}$$

After opening the spin structure of the diagrams in Fig. 4 we arrive at the following set of equations for t_0 , t_0^h , $t_0^{(1)}$ and $t_0^{(2)}$ (for brevity we omit the dependence of the ver-

tices on \vec{n} , ω and \vec{q}), cf. Eq. (55),

$$t_0 - t_0^\omega = \left\langle \int d\Phi_T \Gamma_0^\omega \left[G_+ G_- t_0 - F_+ F_- t_0^h - G_+ F_- t_0^{(1)} - F_+ G_- t_0^{(2)} \right] \right\rangle_{\vec{n}'}, \tag{191a}$$

$$t_0^h - t_0^{h\omega} = \left\langle \int d\Phi_T \Gamma_0^\omega \left[G_+^h G_-^h t_0^h - F_+ F_- t_0 - F_+ G_-^h t_0^{(1)} - G_+^h F_- t_0^{(2)} \right] \right\rangle_{\vec{n}'}, \tag{191b}$$

$$t_0^{(1)} = -\left\langle \int d\Phi_T \Gamma_0^\xi \left[G_+ G_-^h t_0^{(1)} - F_+ F_- t_0^{(2)} + G_+ F_- t_0 + F_+ G_-^h t_0^h \right] \right\rangle_{\vec{n}'}, \tag{191c}$$

$$t_0^{(2)} = -\left\langle \int d\Phi_T \Gamma_0^\xi \left[G_+^h G_- t_0^{(2)} - F_+ F_- t_0^{(1)} + F_+ G_- t_0 + G_+^h F_- t_0^h \right] \right\rangle_{\vec{n}'}. \tag{191d}$$

The similar set of equations for 3-vector vertices \tilde{t}_1 , \tilde{t}_1^h , $\tilde{t}_1^{(1)}$ and $\tilde{t}_1^{(2)}$ is written with the only differences that $\Gamma_0^{\omega, \xi}$ is replaced by $\Gamma_1^{\omega, \xi}$ and in front of all terms with t^h the sign must be changed, cf. Eq. (56). For the sake of convenience we introduce brief notations, e.g.,

$$\int d\Phi_T G_+^h F_- = G_+^h \cdot F_-. \tag{192}$$

The details of calculations of these products within the Matsubara technique are given in Ref. [95]. For instance the useful relations introduced in Ref. [17]

$$\begin{aligned}G_+ \cdot G_-^h &= G_+^h \cdot G_-, \quad G_+ \cdot F_- = -F_+ \cdot G_-, \\ F_+ \cdot G_-^h &= -G_+^h \cdot F_-, \end{aligned}\tag{193}$$

are recovered for arbitrary temperature. From (191) we can immediately find relations between the vertices $t_0^{(1)}$ and $t_0^{(2)}$. Taking the sum of Eqs. (191c) and (191d) and making use of Eq. (193) we obtain the homogeneous equation for the sum $t_0^{(1)} + t_0^{(2)}$,

$$\begin{aligned}t_0^{(1)} + t_0^{(2)} &= -\left\langle \int d\Phi_T \Gamma^\xi \left[G_+ G_-^h - F_+ F_- \right] \right. \\ &\quad \left. \times (t_0^{(1)} + t_0^{(2)}) \right\rangle_{\vec{n}'},\end{aligned}\tag{194}$$

which implies

$$t_0^{(1)} + t_0^{(2)} = 0 \tag{195}$$

for frequencies relevant for PBF processes, which we will consider below. The latter relation justifies the parameterization of the full in-medium vertices used below in Eqs. (202b) – (202d). The same relation is valid for $\tilde{t}_1^{(1)}$ and $\tilde{t}_1^{(2)}$ vertices.

In their original paper [15] A.I. Larkin and A.B. Migdal presented Eqs. (191) in somewhat different form. They

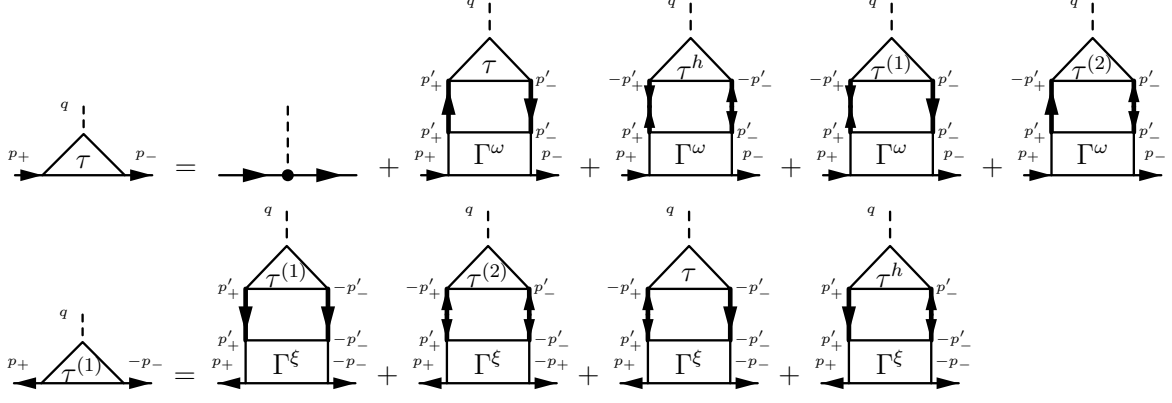


FIG. 4: Equations for the dressed vertices $\hat{\tau}$ and $\hat{\tau}^{(1)}$ from Eq. (46) in the momentum representation.

noted that the vertices for the holes, t^h , can be obtained from the particle vertices with the replacement $\vec{n} \rightarrow -\vec{n}$,

$$t_0^h(\vec{n}, q) = t_0(-\vec{n}, q), \quad \vec{t}_1^h(\vec{n}, q) = \vec{t}_1(-\vec{n}, q). \quad (196)$$

Therefore, one can introduce the operator $\hat{\pi}$, which performs this change of \vec{n} in the vertex

$$\hat{\pi} t_0(\vec{n}, q) = t_0^h(\vec{n}, q), \quad \hat{\pi} \vec{t}_1(\vec{n}, q) = \vec{t}_1^h(\vec{n}, q). \quad (197)$$

Because of relation (195), Eqs. (191c, 191d) reduce to one equation for the vertex

$$\tilde{t}_0 = -t_0^{(1)} = t_0^{(2)}.$$

Analogously we introduce $\tilde{\vec{t}}_1 = -\vec{t}_1^{(1)} = \vec{t}_1^{(2)}$. Then four Eqs. (191) for scalar vertices '0' and four equations for 3-vector vertices \vec{t}_1 can be cast in terms of four equations

$$t_0 - t_0^\omega = \langle \Gamma_0^\omega (L(\hat{\pi}) t_0 + M \tilde{t}_0) \rangle_{\vec{n}'}, \quad (198a)$$

$$\tilde{t}_0 = -\langle \Gamma_0^\xi ((N + A_0) \tilde{t}_0 + O(\hat{\pi}) t_0) \rangle_{\vec{n}'}, \quad (198b)$$

$$\vec{t}_1 - \vec{t}_1^\omega = \langle \Gamma_1^\omega (L(-\hat{\pi}) \vec{t}_1 + M \tilde{\vec{t}}_1) \rangle_{\vec{n}'}, \quad (198c)$$

$$\tilde{\vec{t}}_1 = -\langle \Gamma_1^\xi ((N + A_0) \tilde{\vec{t}}_1 + O(-\hat{\pi}) \vec{t}_1) \rangle_{\vec{n}'}. \quad (198d)$$

We shall call this set of equations *the Larkin-Migdal equations*. In Ref. [15] these four equations are further reduced to only two equations with the help of the operator \hat{P} (defined by Eq. (31) in Ref. [15]), which includes additionally the change of the sign between Eqs. (198a) and (198c) and between Eqs. (198b) and (198d). Functions L , M , N , and O are defined as in Ref. [15]:

$$\begin{aligned} L(\vec{n}, q; \hat{\pi}) &= G_+ \cdot G_- - F_+ \cdot F_- \hat{\pi}, \\ M(\vec{n}, q) &= G_+ \cdot F_- - F_+ \cdot G_-, \\ N(\vec{n}, q) &= G_+ \cdot G_-^h + F_+ \cdot F_- - A_0, \\ O(\vec{n}, q; \hat{\pi}) &= -G_+ \cdot F_- - F_+ \cdot G_-^h \hat{\pi}. \end{aligned} \quad (199)$$

We emphasize that Eqs. (198) are valid at arbitrary temperature. The temperature dependence is hidden in the convolutions of the Green's functions (199). In Ref. [15] the latter ones were calculated explicitly only for $T = 0$ using the method of Ref. [143]. The extension to $T \neq 0$ is done in Ref. [95], see Appendix D.

VII. CURRENT-CURRENT CORRELATORS FOR THE PBF PROCESS $n \rightarrow n + \nu \bar{\nu}$

Below we demonstrate how to apply the above results to calculate the neutrino emissivity in the PBF process $n \rightarrow n + \nu \bar{\nu}$ from the superfluid neutron star interior. To perform this calculation [94, 95] we start with the diagrammatic presentation of the current-current correlator (57) for weak neutral currents, where enter the vertices shown in Fig. 4, calculated the previous section. In Eqs. (57), (58) we used the contour Green's functions, whereas now we will treat the same Eq. (57) for the retarded self-energy. We use the Matsubara technique at $T \neq 0$ and then perform appropriate analytic continuation to obtain the retarded self-energy. Thus we recover the RPA current-current correlator.

The bare vertices generated by the weak currents (70) are equal to

$$\hat{\tau}_V^\omega(\vec{n}, q) = g_V (\tau_{V,0}^\omega l_0 - \vec{\tau}_{V,1}^\omega \vec{l}) \hat{1}, \quad (200a)$$

$$\tau_{V,0}^\omega = \frac{e_V}{a}, \quad \vec{\tau}_{V,1}^\omega = \frac{e_V}{a} \vec{v},$$

$$\hat{\tau}_A^\omega(\vec{n}, q) = -g_A (\vec{\tau}_{A,1}^\omega \vec{\sigma} l_0 - \tau_{A,0}^\omega \vec{\sigma} \vec{l}), \quad (200b)$$

$$\tau_{A,0}^\omega = \frac{e_A}{a}, \quad \vec{\tau}_{A,1}^\omega = \frac{e_A}{a} \vec{v}.$$

Here e_V and e_A are effective charges of the vector and axial-vector currents. For the vector current $e_V = 1$ and for the axial-vector current $e_A \simeq 0.8 - 0.95$, as it follows from studies of the Gamov-Teller transitions in nuclei, see Refs. [6, 141, 144] and references therein. The corre-

sponding vertices for holes are defined as

$$\begin{aligned}\hat{\tau}_V^{\omega,h}(\vec{n},q) &= [\hat{\tau}_V^{\omega}(-\vec{n},q)]^T \\ &= g_V (\tau_{V,0}^{\omega} l_0 + \vec{\tau}_{V,1}^{\omega} \vec{l}) \sigma_0,\end{aligned}\quad (201a)$$

$$\begin{aligned}\hat{\tau}_A^{\omega,h}(\vec{n},q) &= [\hat{\tau}_A^{\omega}(-\vec{n},q)]^T \\ &= -g_A (-\vec{\tau}_{A,1}^{\omega} \vec{\sigma}^T l_0 - \tau_{A,0}^{\omega} \vec{\sigma}^T \vec{l}).\end{aligned}\quad (201b)$$

We used here explicitly that $\tau_{a,0}^{\omega}(-p,q) = \tau_{a,0}^{\omega}(p,q)$ and $\vec{\tau}_{a,1}^{\omega}(-p,q) = -\vec{\tau}_{a,1}^{\omega}(p,q)$.

The structure of the full vertices reads

$$\begin{aligned}\hat{\tau}_V &= g_V (\tau_{V,0} l_0 - \vec{\tau}_{V,1} \vec{l}) \sigma_0, \\ \hat{\tau}_V^h &= g_V (\tau_{V,0}^h l_0 - \vec{\tau}_{V,1}^h \vec{l}) \sigma_0,\end{aligned}\quad (202a)$$

$$\begin{aligned}\hat{\tau}_V^{(1)} &= (\hat{\tau}_V^{(2)})^\dagger \\ &= -g_V (\vec{\tau}_{V,0} l_0 - \vec{\tau}_{V,1} \vec{l}) i \sigma_2,\end{aligned}\quad (202b)$$

$$\begin{aligned}\hat{\tau}_A &= -g_A (\vec{\tau}_{A,1} \vec{\sigma} l_0 - \tau_{A,0} \vec{\sigma} \vec{l}), \\ \hat{\tau}_A^h &= -g_A (\vec{\tau}_{A,1}^h \vec{\sigma}^T l_0 - \tau_{A,0}^h \vec{\sigma}^T \vec{l}),\end{aligned}\quad (202c)$$

$$\begin{aligned}\hat{\tau}_A^{(1)} &= (\hat{\tau}_A^{(2)})^\dagger \\ &= g_A (\vec{\tau}_{A,1} \vec{\sigma} l_0 - \vec{\tau}_{A,0} \vec{\sigma} \vec{l}) i \sigma_2.\end{aligned}\quad (202d)$$

We rewrite the retarded current-current correlator corresponding to (72) as

$$\Sigma_{[\text{nuc}],\mu\nu}^R l^\mu l^\nu = \chi(q) = \chi_V(q) + \chi_A(q), \quad (203)$$

where the contributions of vector and axial currents are

$$\begin{aligned}\chi_V(q) &= g_V^2 \langle (\tau_{V,0}^{\omega} l_0 - \vec{\tau}_{V,1}^{\omega} \vec{l}) \\ &\quad \times (l_0 \chi_{V,0}(\vec{n},q) - \vec{\chi}_{V,1}(\vec{n},q) \vec{l}) \rangle_{\vec{n}},\end{aligned}\quad (204a)$$

$$\begin{aligned}\chi_A(q) &= g_A^2 \langle (\vec{\tau}_{A,1}^{\omega} l_0 - \tau_{A,0}^{\omega} \vec{l}) \\ &\quad \times (l_0 \vec{\chi}_{A,1}(\vec{n},q) - \chi_{A,0}(\vec{n},q) \vec{l}) \rangle_{\vec{n}}\end{aligned}\quad (204b)$$

with the scalar and vector response functions

$$\begin{aligned}\chi_{a,0}(\vec{n},q) &= \int d\Phi_T (G_+ G_- \tau_{a,0} - F_+ F_- \tau_{a,0}^h \\ &\quad + (G_+ F_- - F_+ G_-) \tilde{\tau}_{a,0}),\end{aligned}\quad (205a)$$

$$\begin{aligned}\vec{\chi}_{a,1}(\vec{n},q) &= \int d\Phi_T (G_+ G_- \vec{\tau}_{a,1} + F_+ F_- \vec{\tau}_{a,1}^h \\ &\quad + (G_+ F_- - F_+ G_-) \tilde{\tau}_{a,0}).\end{aligned}\quad (205b)$$

Then we apply the Larkin-Migdal Eqs. (198) for the case of the weak-current vertices (201,202). For the vector-current vertices we use Eqs. (198a,198b) and for the axial-vector-current vertices, Eqs. (198c,198d). Then we separate the parts proportional to the scalar l_0 and the vector \vec{l} and obtain altogether eight equations for the vector and axial-vector current vertices. These sets of equations are cast in the following form [94],

$$\tau_{a,0}(\vec{n},q) = \tau_{a,0}^{\omega}(\vec{n},q) + \langle \Gamma_a^{\omega}(\vec{n},\vec{n}') [L(\vec{n}',q; \hat{P}_{a,0}) \tau_{a,0}(\vec{n}',q) + M(\vec{n}',q) \tilde{\tau}_{a,0}(\vec{n}',q)] \rangle_{\vec{n}'}, \quad (206a)$$

$$\tilde{\tau}_{a,0}(\vec{n},q) = - \langle \Gamma_a^{\xi}(\vec{n},\vec{n}') [(N(\vec{n}',q) + A_0) \tilde{\tau}_{a,0}(\vec{n}',q) + O(\vec{n}',q; \hat{P}_{a,0}) \tau_{a,0}(\vec{n}',q)] \rangle_{\vec{n}'}, \quad (206b)$$

$$\vec{\tau}_{a,1}(\vec{n},q) = \vec{\tau}_{a,1}^{\omega}(\vec{n},q) + \langle \Gamma_a^{\omega}(\vec{n},\vec{n}') [L(\vec{n}',q; \hat{P}_{a,1}) \vec{\tau}_{a,1}(\vec{n}',q) + M(\vec{n}',q) \tilde{\vec{\tau}}_{a,1}(\vec{n}',q)] \rangle_{\vec{n}'}, \quad (206c)$$

$$\tilde{\vec{\tau}}_{a,1}(\vec{n},q) = - \langle \Gamma_a^{\xi}(\vec{n},\vec{n}') [(N(\vec{n}',q) + A_0) \tilde{\vec{\tau}}_{a,1}(\vec{n}',q) + O(\vec{n}',q; \hat{P}_{a,1}) \vec{\tau}_{a,1}(\vec{n}',q)] \rangle_{\vec{n}'}, \quad (206d)$$

where $a = V$ for the vector and $a = A$ for the axial-vector weak currents. The notation for the effective interaction is $\Gamma_V^{\omega,\xi} = \Gamma_0^{\omega,\xi}$ and $\Gamma_A^{\omega,\xi} = \Gamma_1^{\omega,\xi}$. Operators $\hat{P}_{a,i}$ are defined as follows

$$\hat{P}_{a,i} = (-1)^i P_{a,i} \hat{\pi}, \quad i = 0, 1 \quad (207)$$

with parameters

$$P_{V,0} = 1, P_{V,1} = -1, P_{A,0} = -1, P_{A,1} = 1, \quad (208)$$

which are the eigenvalues of the operators $\hat{P}_{a,i}$, acting on the bare vertices

$$\hat{P}_{a,0} \tau_{a,0}^{\omega} = P_{a,0} \tau_{a,0}^{\omega}, \quad \hat{P}_{a,1} \vec{\tau}_{a,1}^{\omega} = P_{a,1} \vec{\tau}_{a,1}^{\omega}. \quad (209)$$

To proceed, let us for simplicity assume that Γ_a^{ω} and Γ_a^{ξ} contain only zero-th Legendre harmonics. From (206b)

we find

$$\tilde{\tau}_{a,0}(q) = -\eta_a^{\xi} \frac{\langle O(\vec{n},q; P_{a,0}) \rangle_{\vec{n}}}{\langle N(\vec{n},q) \rangle_{\vec{n}}} \tau_{a,0}(q), \quad (210)$$

where

$$\eta_a^{\xi} = \frac{\Gamma_a^{\xi} \langle N(\vec{n},q) \rangle_{\vec{n}}}{1 + \Gamma_a^{\xi} [A_0 + \langle N(\vec{n},q) \rangle_{\vec{n}}]}. \quad (211)$$

For the channel $a = V$, for which the gap equation $1 = -A_0 \Gamma_V^{\xi}$ is valid, we obtain $\eta_V^{\xi} = 1$. For the other channel, $\eta_A^{\xi} \neq 1$. Substituting Eq. (210) in Eq. (206a) we obtain

$$\begin{aligned}\tau_{a,0}(q) &= \gamma_a(q; P_{a,0}) \tau_{a,0}^{\omega}, \\ \gamma_a^{-1}(q; P) &= 1 - \Gamma_a^{\omega} \langle \mathcal{L}_a(\vec{n},q; P) \rangle_{\vec{n}},\end{aligned}\quad (212)$$

where we introduced the notation

$$\mathcal{L}_a(\vec{n}, q; P) = L(\vec{n}, q; P) - \eta_a^\xi \frac{\langle O(\vec{n}, q; P) \rangle_{\vec{n}}}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} M(\vec{n}, q). \quad (213)$$

Solving the second pair of the Larkin-Migdal equations (206c, 206d) we first note that for the constants Γ_a^ω and Γ_a^ξ the angular averages on the right-hand sides of equations do not depend on \vec{n} . Therefore, the component of the bare vertex proportional to \vec{v} is not renormalized in medium. However, in view of the identity

$$\langle f(\vec{n}, \vec{q}) \vec{n} \rangle_{\vec{n}} = \langle f(\vec{n}, \vec{q}) (\vec{n} \vec{q}) \rangle_{\vec{n}} \vec{q} / \vec{q}^2 \quad (214)$$

valid for an arbitrary scalar function f of \vec{n} and \vec{q} , the full vertices gain a component proportional to \vec{q} . Thus, we decompose 3-vectors $\vec{\tau}_{a,1}(\vec{n}, q)$ and $\vec{\tilde{\tau}}_{a,1}(\vec{n}, q)$ into the parts proportional to the \vec{n} and $\vec{n}_q = \vec{q}/|\vec{q}|$ vectors and introduce new scalar form-factors

$$\begin{aligned} \vec{\tau}_{a,1}(\vec{n}, q) &= \tau_{a,1}^\omega \vec{n} + \tau_{a,1}^{(q)}(q) \vec{n}_q, \\ \vec{\tilde{\tau}}_{a,1}(\vec{n}, q) &= \tilde{\tau}_{a,1}^{(q)}(q) \vec{n}_q \end{aligned} \quad (215)$$

with the bare vertex $\tau_{a,1}^\omega = (\vec{n} \vec{\tau}_{a,1}^\omega)$. Action of the operator $\hat{P}_{a,1}$ on the vertices (215) is given by

$$\begin{aligned} \hat{P}_{a,1} \vec{\tau}_{a,1}(\vec{n}, q) &= P_{a,1} \tau_{a,1}^\omega \vec{n} + (-P_{a,1}) \vec{n}_q \tau_{a,1}^{(q)}(q), \\ \hat{P}_{a,1} \vec{\tilde{\tau}}_{a,1}(\vec{n}, q) &= (-P_{a,1}) \vec{n}_q \tilde{\tau}_{a,1}^{(q)}(q). \end{aligned}$$

Then, from Eq. (206d) we recover

$$\begin{aligned} \tilde{\tau}_{a,1}^{(q)} &= -\eta_a^\xi \frac{\langle O(\vec{n}, q; -P_{a,1}) \rangle_{\vec{n}}}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} \tau_{a,1}^{(q)} \\ &- \eta_a^\xi \frac{\langle O(\vec{n}, q; P_{a,1}) (\vec{n} \vec{n}_q) \rangle_{\vec{n}}}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} \tau_{a,1}^\omega. \end{aligned} \quad (216)$$

From Eq. (206c), substituting there Eq. (216), we find

$$\begin{aligned} \tau_{a,1}^{(q)} &= \gamma_a(q; -P_{a,1}) \Gamma_a^\omega \\ &\times \langle \tilde{\mathcal{L}}_a(\vec{n}, q; P_{a,1}) (\vec{n} \cdot \vec{n}_q) \rangle_{\vec{n}} \tau_{a,1}^\omega. \end{aligned} \quad (217)$$

Here we introduced the quantity

$$\begin{aligned} \tilde{\mathcal{L}}_a(\vec{n}, q; P) &= L(\vec{n}, q; P) \\ &- \eta_a^\xi \frac{\langle M(\vec{n}, q) \rangle_{\vec{n}}}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} O(\vec{n}, q; P) \end{aligned} \quad (218)$$

and used the identity

$$\langle \tilde{\mathcal{L}}_a(\vec{n}, q; P_{a,1}) \rangle_{\vec{n}} = \langle \mathcal{L}_a(\vec{n}, q; P_{a,1}) \rangle_{\vec{n}}, \quad (219)$$

which allows to use for the vector vertices the same function γ_a as for the scalar vertex.

In terms of the loop-functions (199) the response functions (205a, 205b) can be expressed as

$$\begin{aligned} \chi_{a,0}(\vec{n}, q) &= L(\vec{n}, q; \hat{P}_{a,0}) \tau_{a,0}(\vec{n}, q) + M(\vec{n}, q) \tilde{\tau}_{a,0}(\vec{n}, q), \\ \bar{\chi}_{a,1}(\vec{n}, q) &= L(\vec{n}, q; \hat{P}_{a,1}) \vec{\tau}_{a,1}(\vec{n}, q) + M(\vec{n}, q) \vec{\tilde{\tau}}_{a,1}(\vec{n}, q). \end{aligned}$$

Using solutions (210) and (212), for the scalar vertices we find

$$\chi_{a,0}(\vec{n}, q) = \gamma_a(q; P_{a,0}) \tau_{a,0}^\omega \mathcal{L}(\vec{n}, q; P_{a,0}). \quad (220)$$

With the help of Eq. (215) we construct

$$\begin{aligned} \bar{\chi}_{a,1}(\vec{n}, q) &= L(\vec{n}, q; P_{a,1}) \vec{n} \tau_{a,1}^\omega \\ &+ L(\vec{n}, q; -P_{a,1}) \vec{n}_q \tau_{a,1}^{(q)}(\vec{n}, q) \\ &+ M(\vec{n}, q) \vec{n}_q \tilde{\tau}_{a,1}^{(q)}(\vec{n}, q). \end{aligned} \quad (221)$$

Using solutions (216) and (217) for the three-vector vertices we obtain

$$\begin{aligned} \bar{\chi}_{a,1}(\vec{n}, q) &= \left(L(\vec{n}, q; P_{a,1}) \vec{n} \right. \\ &- M(\vec{n}, q) \eta_a^\xi \frac{\langle O(\vec{n}, q; P_{a,1}) (\vec{n} \vec{n}_q) \rangle_{\vec{n}}}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} \vec{n}_q \left. \right) \tau_{a,1}^\omega \\ &+ \mathcal{L}_a(\vec{n}, q; -P_{a,1}) \gamma_a(q; -P_{a,1}) \\ &\times \Gamma_a^\omega \langle \tilde{\mathcal{L}}_a(\vec{n}, q; P_{a,1}) (\vec{n} \vec{n}_q) \rangle_{\vec{n}} \vec{n}_q \tau_{a,1}^\omega, \end{aligned} \quad (222)$$

and, then, rewrite it as follows

$$\begin{aligned} \bar{\chi}_{a,1}(\vec{n}, q) &= \tilde{\tau}_{a,1}^\omega \gamma_a(q; -P_{a,1}) \mathcal{L}_a(\vec{n}, q; P_{a,1}) \\ &+ \delta \bar{\chi}_{a,1}(\vec{n}, q), \\ \delta \bar{\chi}_{a,1}(\vec{n}, q) &= \eta_a^\xi \frac{M(\vec{n}, q)}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} \langle O(\vec{n}', q; P_{a,1}) (\vec{n} - \vec{n}') \rangle_{\vec{n}'} \tau_{a,1}^\omega \\ &+ \gamma_a(q; -P_{a,1}) \tau_{a,1}^\omega \Gamma_a^\omega \left(\mathcal{L}_a(\vec{n}, q; -P_{a,1}) \right. \\ &\times \langle \tilde{\mathcal{L}}_a(\vec{n}, q; P_{a,1}) \vec{n} \rangle_{\vec{n}} \\ &\left. - \mathcal{L}_a(\vec{n}, q; P_{a,1}) \vec{n} \langle \tilde{\mathcal{L}}_a(\vec{n}, q; -P_{a,1}) \rangle_{\vec{n}} \right). \end{aligned} \quad (223)$$

Relations

$$\begin{aligned} \langle \omega \chi_{V,0} - \vec{q} \vec{\chi}_{V,1} \rangle_{\vec{n}} &= 0, \\ \Im \langle (\vec{q} \vec{v}) (\omega \chi_{V,0} - \vec{q} \vec{\chi}_{V,1}) \rangle_{\vec{n}} &= 0 \end{aligned} \quad (224)$$

proved in Ref. [95] ensure the transversality of the polarization tensor for the weak vector current. Thus, we have demonstrated that the retarded self-energy given by the RPA set of diagrams (58) with the vertices shown by Fig. 4 complies with the vector current conservation.

Below we exploit expressions for the following averages

$$\langle \chi_{a,0}(\vec{n}, q) \rangle_{\vec{n}} = \gamma_a(q; P_{a,0}) \tau_{a,0}^\omega \times \langle \mathcal{L}(\vec{n}, q; P_{a,0}) \rangle_{\vec{n}}, \quad (225a)$$

$$\langle \chi_{a,0}(\vec{n}, q) (\vec{q} \vec{v}) \rangle_{\vec{n}} = \gamma_a(q; P_{a,0}) \tau_{a,0}^\omega \times \langle \mathcal{L}(\vec{n}, q; P_{a,0}) (\vec{q} \vec{v}) \rangle_{\vec{n}}, \quad (225b)$$

$$\langle \vec{q} \vec{\chi}_{a,1}(\vec{n}, q) \rangle_{\vec{n}} = \gamma_a(q; -P_{a,1}) \times \langle \tilde{\mathcal{L}}_a(\vec{n}, q; P_{a,1}) (\vec{\tau}_{a,1}^\omega \vec{q}) \rangle_{\vec{n}}. \quad (225c)$$

$$\begin{aligned} \langle \vec{v} \vec{\chi}_{a,1}(\vec{n}, q) \rangle_{\vec{n}} &= \langle L(\vec{n}, q; P_{a,1}) (\vec{v} \vec{\tau}_{a,1}^\omega) \rangle_{\vec{n}} \\ &- \langle M(\vec{n}, q) (\vec{v} \vec{n}_q) \rangle_{\vec{n}} \eta_a^\xi \frac{\langle O(\vec{n}, q; P_{a,1}) (\vec{\tau}_{a,1}^\omega \vec{n}_q) \rangle_{\vec{n}}}{\langle N(\vec{n}, q) \rangle_{\vec{n}}} \\ &+ \gamma_a(q; -P_{a,1}) \Gamma_a^\omega \langle \mathcal{L}_a(\vec{n}, q; -P_{a,1}) (\vec{v} \vec{n}_q) \rangle_{\vec{n}} \\ &\times \langle \tilde{\mathcal{L}}_a(\vec{n}, q; P_{a,1}) (\vec{\tau}_{a,1}^\omega \vec{n}_q) \rangle_{\vec{n}}. \end{aligned} \quad (225d)$$

In the region of the $1S_0$ neutron pairing in neutron stars occurs at low densities ($< n_0$) and the nucleons are non-relativistic, $v_F \ll 1$. Expanding \mathcal{L} and L at small v_F , see Refs. [94, 95], we find that the correlation functions γ_a differ from unity only in the second order in $v_F |\vec{q}|$, i.e.

$$\gamma_a(q; P) \approx 1 + O(\Gamma_a^\omega a^2 \rho \vec{q}^2 v_F^2 / \omega^2). \quad (226)$$

We obtain that in the expression induced by the vector currents both scalar and vector components, (225a) and (225d), contribute at the order v_F^4 ,

$$\begin{aligned} \Im \langle \chi_{V,0}(\vec{n}, q) \rangle &\approx -\frac{4 \vec{q}^4 v_F^4}{45 \omega^4} e_V a \rho \Im g_T(0, \omega, 0), \\ \Im \langle \vec{v} \vec{\chi}_{V,1}(\vec{n}, q) \rangle &\approx -\frac{2 \vec{q}^2 v_F^4}{9 \omega^2} e_V a \rho \Im g_T(0, \omega, 0). \end{aligned}$$

Working in the leading order in v_F we have to put $\gamma_V \rightarrow 1$ in view of Eq. (226). The function g_T is introduced in Appendix D. From (D6) it follows that

$$\Im g_T(0, \omega, 0) = -\frac{2\pi \Delta^2 \theta(\omega - 2\Delta)}{\omega \sqrt{\omega^2 - 4\Delta^2}} \left(1 - 2n\left(\frac{\omega}{2T}\right)\right).$$

VIII. NEUTRINO EMISSIVITY IN THE NEUTRON PBF PROCESS

Once the averages (225) are known, expressions (66) and (76) solve the problem of the neutrino emission from the superfluid neutron matter via the PBF reactions. The emissivity is given by

$$\begin{aligned} \varepsilon_{\nu\nu} &= \varepsilon_{\nu\nu,V} + \varepsilon_{\nu\nu,A}, \\ \varepsilon_{\nu\nu,a} &= \frac{G^2}{48\pi^4} \int_0^\infty d\omega \int_0^\omega d|\vec{q}| \frac{\omega \vec{q}^2}{e^{\omega/T} - 1} K_a(q), \end{aligned} \quad (227)$$

where K_a is the correlator function summed over the lepton spins and integrated over the lepton phase space

$$K_a(q) = \frac{3}{4\pi} \int d\Phi_l \Im \sum_{\text{spin}} \chi_a^{\mu\nu}(q). \quad (228)$$

The integration over the lepton phase space can be performed analytically with the help of Eqs. (B2) and (224), and we find, see Ref. [94] for details,

$$\begin{aligned} K_V(q) &= \frac{e_V g_V^2}{a} (\vec{q}^2 - \omega^2) \\ &\times \Im \langle \chi_{V,0}(\vec{n}, q) - \vec{v} \vec{\chi}_{V,1}(\vec{n}, q) \rangle_{\vec{n}}, \end{aligned} \quad (229)$$

$$\begin{aligned} K_A(q) &= \frac{e_A g_A^2}{a} \Im [\vec{q}^2 \langle \vec{v} \vec{\chi}_{A,1}(\vec{n}, q) \rangle_{\vec{n}} \\ &+ (3\omega^2 - 2\vec{q}^2) \langle \chi_{A,0}(\vec{n}, q) \rangle_{\vec{n}} \\ &- \omega \langle \vec{q} \vec{\chi}_{A,1}(\vec{n}, q) \rangle_{\vec{n}} - \omega \langle (\vec{q} \vec{v}) \chi_{A,0}(\vec{n}, q) \rangle_{\vec{n}}]. \end{aligned} \quad (230)$$

For the neutron PBF emissivity on the vector current we obtain [94] (for one neutrino flavor)

$$\epsilon_{\nu\nu,V}^{\text{nPBF}} \simeq \epsilon_{\nu\nu}^{(0n)} g_V^2 e_V^2 \frac{4}{81} v_{F,n}^4, \quad g_V^2 e_V^2 = 1. \quad (231)$$

Here

$$\begin{aligned} \epsilon_{\nu\nu}^{(0n)} &= \frac{4\rho G^2 \Delta_n^7}{15\pi^3} I\left(\frac{\Delta}{T}\right), \\ I(z) &= \int_1^\infty \frac{dy y^5}{\sqrt{y^2 - 1}} \frac{1}{(e^{zy} + 1)^2} \end{aligned} \quad (232)$$

is the result derived previously in Ref. [35, 68, 91].

Now let us turn to neutrino emissivity induced by the axial-vector current. In the expansion $v_F \ll 1$ the leading term contributing to the emissivity is of the order v_F^2 . Keeping only the leading terms we cast Eq. (230) as

$$\begin{aligned} K_A &\approx -g_A^2 e_A^2 \rho v_F^2 \vec{q}^2 \\ &\times \left[1 + \left(1 - \frac{2}{3} \frac{\vec{q}^2}{\omega^2}\right) - \frac{2}{3}\right] \Im g_T(0, \omega, 0). \end{aligned} \quad (233)$$

The correlation factors γ_a contribute at the sub-leading order $\sim v_F^4$, therefore we neglected these terms in the approximate expression (233). As the result, the neutron PBF emissivity induced by the axial-vector current is (for one neutrino flavor) given by

$$\epsilon_{\nu\nu,A}^{\text{nPBF}} \simeq \left(1 + \frac{11}{21} - \frac{2}{3}\right) g_A^2 e_A^2 v_{F,n}^2 \epsilon_{\nu\nu}^{(0n)}. \quad (234)$$

The resulting emissivity is the sum of contributions (231) and (234). We stress that Eqs. (231) and (234) are approximate expressions obtained in the leading order in v_F . General result looks more cumbersome but it is easily recovered with the help of Eqs. (225). The latter equations are derived in [95] and here at arbitrary temperature.

IX. CONCLUSION

A.I. Larkin and A.B. Migdal extended the Landau's Fermi-liquid theory onto superfluid systems. In this

paper we re-formulated their approach for systems out of equilibrium. For that we used Schwinger-Kadanoff-Baym-Keldysh formalism. Important improvements of the Larkin-Migdal approach compared to the Nambu-Gorkov one are that the former approach allows to deal with strong interactions different in the particle-hole and particle-particle channels. These achievements have been used by A.J. Leggett who generalized the Larkin-Migdal approach to describe strongly interacting fermion superfluids at finite temperatures and applied it to description of superfluid ^3He . He used Matsubara diagram technique. The use of the Schwinger-Keldysh diagram technique allows to consider variety of non-equilibrium problems. In application to nucleon systems, in general, the considered in this paper formalism can be applied to the pairing in the states with an arbitrary angular momentum; it operates with various forms of a nucleon-nucleon interaction: scalar, spin-spin, spin-orbit and tensor interactions. As argued by A.B. Migdal the tensor forces mediated by the pion exchange should enhance (pion softening) with increase of the nucleon density. Inclusion of this effect might be important in the case of the P -pairing in neutron star interiors.

We considered the neutrino radiation from a finite piece of the nuclear matter bearing in mind the problem of the neutron-star cooling. We used optical theorem formalism formulated in terms of closed diagrams with the full fermion and boson Green's functions and the full nucleon-nucleon interaction. The series of the diagrams is constructed with respect to the number, N , of the full G^{-+} fermion Green's functions. For simplification we considered a system which evolves slowly in time and has small spatial gradients. This allowed us to perform the gradient expansion after the Wigner transformation and keep only gradient-independent terms in calculations of reaction rates. We demonstrated that in order to exactly satisfy the vector current conservation in the nucleon pair breaking and formation processes it is not sufficient to include only one $N = 1$ term of the series, rather one needs to re-sum the RPA series including multi-piece diagrams. (The multi-piece diagrams decay in more than two pieces, being cut through $(-+)$, $(+-)$ lines). This demonstration shows, how one should separate one-nucleon, two-nucleon, etc. processes, in accordance with exact conservation law of the vector current. Comparison of the RPA Σ^{-+} self-energy with the $N = 1$ contribution shows the accuracy with which one may deal, using only one $N = 1$ diagram.

Then we demonstrated how the developed formalism allows to calculate neutrino emissivity from the piece of a warm nucleon matter in presence of the nucleon pairing. As simplest example we calculated neutrino emissivity in the neutron pair breaking and formation processes. These processes are of one-nucleon origin. To simplify consideration, we focused on the case of the ordinary $1S_0$ pairing of neutrons. More difficult is to calculate the emissivity of the two-nucleon ($N = 2$) processes, and $N \geq 3$ processes in the presence of pairing. The

existing nowadays results for the reaction rates in nucleon systems with pairing are based on the so-called R phase-space suppression factors used to reduce the production rates calculated without pairing. Such an approach can be used only for rough estimations. The formalism formulated in the present paper is fully suited to properly perform the calculations. It is also interesting to search for new processes which might be open in the non-equilibrium and equilibrium medium because of the interaction between different reaction channels. These questions require a separate consideration. In the present paper we focused on the neutrino radiation problem. However, the white body radiation of other quanta can be considered in similar way.

The calculated rate Σ^{-+} can be considered as the gain term in the generalized kinetic equation for the virtual W/Z boson or for the anti-neutrino. For consistency then one needs to include first-order gradient memory terms into the collision term.

Another important question is how to go beyond the quasi-particle approximation for fermions in strongly interacting fermion systems with pairing.

Within the quasi-particle approximation for fermions, the formalism based on the Fermi-liquid renormalization is developed. To quantify the results it remains to know the Landau-Migdal parameters. For the problems under consideration one needs to know them as functions of the density, isospin composition, frequency and momentum. The information extracted from analysis of atomic nucleus experiments is definitely insufficient for these purposes. Existing calculations of the Landau-Migdal parameters are still incomplete. We hope that the present study will motivate further attempts to extract these parameters. In spite of all difficulties, it seems to be the cheapest way to achieve understanding of many new interesting phenomena occurring in the strongly interacting fermion systems in the presence of the pairing.

The direction of the research was shown in the works done in 50th–70th years of the XXth century. The pioneering contribution to the development of the methods of the quantum many-body theory including the problem of fermion pairing belongs to Arkady Migdal.

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Appendix A: Matrix notation

The Schwinger-Keldysh contour in Fig. 1 consists of two branches: time-ordered and anti-time ordered. For a given space-time coordinate x the contour coordinates take two values, x^+ and x^- , depending on the branch

of the contour. The closed real-time contour integration can be written as

$$\int_C dx^C \dots = \int_{t_0}^{\infty} dx^- \dots - \int_{t_0}^{\infty} dx^+ \dots, \quad (\text{A1})$$

where only the time limits are explicitly given and the spatial integration d^3x is assumed. The folding of two two-point functions defined on the contour reads as

$$H(x^i, y^k) = \int dz^C \zeta(x^i, z^C) G(z^C, y^k). \quad (\text{A2})$$

Any two-point function $\zeta = \langle \mathcal{T}_C \hat{A}(x) \hat{B}(y) \rangle$ being a function of two contour variables x and y can be viewed as a matrix, which elements are defined in dependence on their belonging to the branches of the contour

$$\zeta^{ij}(x, y) = \zeta(x^i, y^j), \quad i, j \in \{-, +\}. \quad (\text{A3})$$

For Green's functions this convention produces the following matrices

$$\begin{aligned} iG^{ik}(x, y) &= \begin{pmatrix} iG^{--}(x, y) & iG^{-+}(x, y) \\ iG^{+-}(x, y) & iG^{++}(x, y) \end{pmatrix} \\ &= \begin{pmatrix} \langle \mathcal{T} \hat{\Psi}(x) \hat{\Psi}^\dagger(y) \rangle & \mp \langle \hat{\Psi}^\dagger(y) \hat{\Psi}(x) \rangle \\ \langle \hat{\Psi}(x) \hat{\Psi}^\dagger(y) \rangle & \langle \tilde{\mathcal{T}} \hat{\Psi}(x) \hat{\Psi}^\dagger(y) \rangle \end{pmatrix}, \end{aligned} \quad (\text{A4})$$

where \mathcal{T} and $\tilde{\mathcal{T}}$ are the usual time and anti-time ordering operators and the upper sign is for fermions, the lower one is for bosons. Eq. (A4) implies the following relations between non-equilibrium and usual retarded and advanced Green's functions

$$\begin{aligned} G^R(x, y) &= G^{--}(x, y) - G^{-+}(x, y) \\ &= G^{+-}(x, y) - G^{++}(x, y) \\ &= \Theta(x_0 - y_0) [G^{+-}(x, y) - G^{-+}(x, y)], \\ G^A(x, y) &= G^{--}(x, y) - G^{+-}(x, y) \\ &= G^{-+}(x, y) - G^{++}(x, y) \\ &= \Theta(y_0 - x_0) [G^{-+}(x, y) - G^{+-}(x, y)], \end{aligned} \quad (\text{A5})$$

where $\Theta(x_0 - y_0)$ is the step function of the time difference. The similar relations hold for the self-energies

$$\begin{aligned} \Sigma^R(x, y) &= \Sigma^{--}(x, y) + \Sigma^{-+}(x, y) \\ &= -\Sigma^{+-}(x, y) - \Sigma^{++}(x, y) \\ &= \Theta(x_0 - y_0) [-\Sigma^{+-}(x, y) + \Sigma^{-+}(x, y)], \\ \Sigma^A(x, y) &= \Sigma^{--}(x, y) + \Sigma^{+-}(x, y) \\ &= -\Sigma^{-+}(x, y) - \Sigma^{++}(x, y) \\ &= \Theta(y_0 - x_0) [\Sigma^{+-}(x, y) - \Sigma^{-+}(x, y)]. \end{aligned} \quad (\text{A6})$$

The difference in signs compared to (A5) is due to the fact that Σ includes vertices, and $-$ and $+$ vertices differ by the sign.

In terms of matrices defined as in Eq. (A3) the contour integration in the folding of two functions (A2) turns into

the usual space-time coordinate integration applied to the product of matrix-valued functions

$$H(x^i, y^k) = \sum_{j=+,-} \int dz \zeta^{ij}(x, z) \eta^{ij} G^{jk}(z, y), \quad (\text{A7})$$

where η^{ij} is the diagonal matrix with the elements $\eta^{--} = -\eta^{++} = 1$. It takes into account the extra minus sign of the anti-time-ordered branch of the contour.

Applying hermitian conjugation operation to the Green's function definitions (A4) we obtain the following relations

$$\begin{aligned} [G^{\pm\mp}(x, y)]^\dagger &= -G^{\pm\mp}(y, x), \\ [G^{--}(x, y)]^\dagger &= -G^{++}(y, x), \\ [G^R(x, y)]^\dagger &= G^A(y, x), \end{aligned} \quad (\text{A8})$$

and the analogous ones for the self-energies.

Instead of purely coordinate representation for two-point functions it can be more convenient to perform the Wigner transformation and operate in the mixed coordinate-momentum representation. For any two-point function $\zeta(x, y)$ one introduces the relative coordinate $\xi = x - y$ and the middle coordinate $X = \frac{1}{2}(x + y)$ and makes the Fourier transformation from four-space coordinate ξ to four-momentum p :

$$\zeta^{ij}(X; p) = \int d\xi e^{i p \xi} \zeta^{ij}(X + \xi/2, X - \xi/2) \quad (\text{A9})$$

with $i, j \in \{+, -\}$. The quantities G^{+-} and G^{-+} are called the Wigner densities in the eight dimensional (X, p) phase-space. In the mixed Wigner representation Eqs. (A8) read

$$\begin{aligned} [G^{\pm\mp}(X; p)]^\dagger &= -G^{\pm\mp}(X; p), \\ [G^{--}(X; p)]^\dagger &= -G^{++}(X; p), \\ [G^R(X; p)]^\dagger &= G^A(X; p). \end{aligned} \quad (\text{A10})$$

In particular, these relations imply that the functions $iG^{\pm\mp}$ are always real.

The merit of the Wigner transformation is that it allows to set up an approximation scheme for treating the system being not too far out of equilibrium. For slightly inhomogeneous and slowly evolving systems, the degrees of freedom can be subdivided into rapid and slow ones. Then, the variable ξ relates to rapid and short-ranged microscopic processes and the variable X refers to slow and long-ranged collective motions. A gradient expansion with respect to slow degrees of freedom can be applied since the Wigner transformation converts any convolution of two-point functions into a product of the corresponding Wigner functions plus first-order and higher-

order gradient terms

$$\begin{aligned} & \int d\xi e^{ip\xi} \int dz \zeta(x, z) G(z, y) \\ &= \left[e^{\frac{i}{2}\hbar(\partial_p \partial_{x'} - \partial_x \partial_{p'})} \zeta(X; p) G(X'; p') \right] \Big|_{p'=p, X'=X} \\ &\approx \zeta(X; p) G(X; p) + \frac{i\hbar}{2} \left[\frac{\partial \zeta}{\partial p^\mu} \frac{\partial G}{\partial X_\mu} - \frac{\partial \zeta}{\partial X^\mu} \frac{\partial G}{\partial p_\mu} \right]. \quad (\text{A11}) \end{aligned}$$

Including local and only first-order gradient terms, one derives from the set of the Dyson equations the Kaddanoff-Baym generalized kinetic equation describing slow evolution of slightly spatially inhomogeneous system of particles having non-zero mass widths, see [19, 107]. Although the gradient terms are assumed to be smaller than the local terms they should be kept not only in the Vlasov part of the generalized kinetic equation but also in the collision term giving rise to the memory effects [107].

With the help of the relations (A10) we can prove the convenient relation between the functions \tilde{L}^{ij} introduced in Eq. (99):

$$\begin{aligned} [\tilde{L}^{\pm\mp}(p; q)]^\dagger &= -\tilde{L}^{\pm\mp}(p; q), \\ [\tilde{L}^{--}(p; q)]^\dagger &= -\tilde{L}^{++}(p; q). \quad (\text{A12}) \end{aligned}$$

From Eq. (A5) follows, *inter alia*, that among four quantities, e.g., $G^{\pm\mp}$ and $G^{\pm\pm}$, not all are independent, because of the relation⁸ $G^{++} + G^{--} - G^{+-} - G^{-+} = 0$. The similar holds for self-energies $\Sigma^{++} + \Sigma^{--} + \Sigma^{+-} + \Sigma^{-+} = 0$, see. Eq. (A6). Let us check whether the similar completeness relation holds also for the \tilde{L} functions. Applying Eqs. (A5) and (A6) recursively we find

$$\begin{aligned} i(\tilde{L}^{++} + \tilde{L}^{--} - \tilde{L}^{+-} - \tilde{L}^{-+}) &= \\ &= G_{p+q/2}^R [G_{p-q/2}^A + G_{p-q/2}^{+-} - G_{p-q/2}^{-+}] \\ &+ [G_{p+q/2}^R + G_{p+q/2}^{-+} - G_{p+q/2}^{+-}] G_{p-q/2}^A \\ &= G_{p+q/2}^R G_{p-q/2}^R + G_{p+q/2}^A G_{p-q/2}^A. \quad (\text{A13}) \end{aligned}$$

Note that the last two terms vanish after the integration over p_0 ,

$$\int dp_0 G_{p+q/2}^R G_{p-q/2}^R = \int dp_0 G_{p+q/2}^A G_{p-q/2}^A = 0, \quad (\text{A14})$$

since the poles of both integrated functions lie in one and the same semi-plane of the complex plane — $\Im p_0 < 0$ for the retarded functions and $\Im p_0 > 0$ for the advanced

ones — and the integration contour can be close in the opposite semi-plane. Thus the relation

$$\int dp_0 (\tilde{L}^{++} + \tilde{L}^{--} - \tilde{L}^{+-} - \tilde{L}^{-+}) = 0 \quad (\text{A15})$$

holds for \tilde{L} 's but only after the p_0 -integration.

Appendix B: Lepton tensor

Here we calculate the leptonic tensors entering the reaction probabilities (76,77). We will assume that lepton 1 is massive $\omega_1 = \sqrt{m_1^2 + |\vec{q}_1|^2}$, whereas lepton 2 is massless $\omega_2 = |\vec{q}_2|$. The trace of lepton currents is given by

$$\begin{aligned} \sum_{\text{spin}} (l_\mu l_\nu^\dagger) &= 2 \text{Tr}\{\not{q}_1 \gamma_\mu \not{q}_2 \gamma_\nu (1 - \gamma_5)\} \\ &= 8 (q_{1\mu} q_{2\nu} + q_{2\mu} q_{1\nu} - (q_1 \cdot q_2) g_{\mu\nu} \\ &\quad + i \epsilon_{\alpha\mu\beta\nu} q_1^\alpha q_2^\beta). \end{aligned}$$

The integral over the lepton phase space

$$I^{\mu\nu} = \int \frac{d^3 q_1}{2\omega_1} \frac{d^3 q_2}{2\omega_2} q_1^\mu q_2^\nu \delta^{(4)}(q_1 + q_2 - q)$$

can be calculated in the standard way. The result is as follows:

$$\begin{aligned} I^{\mu\nu} &= \frac{\pi}{24} \left[\left(1 + 2\frac{m_1^2}{q^2}\right) (2q^\mu q^\nu + q^2 g^{\mu\nu}) - 3m_1^2 g^{\mu\nu} \right] \\ &\quad \times \left(1 - \frac{m_1^4}{q^4}\right) \theta(\omega) \theta(q^2 - m_1^2). \quad (\text{B1}) \end{aligned}$$

The lepton tensor in Eq. (76) is given by

$$\begin{aligned} T_{\text{lept}}^{\mu\nu}(q) &= 8 (I^{\mu\nu} + I^{\nu\mu} - g^{\mu\nu} I_\mu^\mu) \\ &= \frac{4\pi}{3} \left[\left(1 + 2\frac{m_1^2}{q^2}\right) (q^\mu q^\nu - q^2 g^{\mu\nu}) + \frac{3}{2} m_1^2 g^{\mu\nu} \right] \\ &\quad \times \left(1 - \frac{m_1^4}{q^4}\right) \theta(\omega) \theta(q^2 - m_1^2). \quad (\text{B2}) \end{aligned}$$

Setting $m_1 = 0$ we obtain expression for the $T_{\text{lept}}^{\mu\nu}(q)$ valid for neutral currents (for $\nu\bar{\nu}$).

For the process occurring on the charged current the reaction rates enters the integral:

$$\begin{aligned} & \int \frac{d^3 q_1}{2\omega_1} \frac{d^3 q_2}{2\omega_2} q_1^\mu q_2^\nu [1 - n_f(q_1 \cdot u)] \delta^{(4)}(q_1 + q_2 - q) \\ &= I^{\mu\nu} - J^{\mu\nu}, \end{aligned}$$

where second term appeared due to the Pauli blocking for the charged leptons (electrons or muons). Here u is a four-vector of a collective motion of the medium, in the co-moving frame $u = (1, 0, 0, 0)$, $q_1 \cdot u = q_1^\mu u_\mu$. Using

⁸ Actually in generalized kinetics only two real quantities are independent, e.g. iG^{-+} and $A = -2\Im G^R$. Other quantities, G^{+-} , G^{--} , G^{++} are expressed through them [107]. In equilibrium only one quantity is independent.

two four-vectors q and u we write the general structure of $J^{\mu\nu}$ as

$$J^{\mu\nu} = A q^\mu q^\nu + B q^\mu u^\nu + C u^\mu q^\nu + D u^\mu u^\nu + E g^{\mu\nu}, \quad (\text{B3})$$

or alternatively as

$$J^{\mu\nu} = C_{qq} t_q^{\mu\nu} + C_{uu} t_u^{\mu\nu} + C_{uq} j_u^\mu j_q^\nu + C_{qu} j_q^\mu j_u^\nu + C_g g^{\mu\nu}, \quad (\text{B4})$$

where we use tensors and four-vectors

$$t_q^{\mu\nu} = q^2 g^{\mu\nu} - q^\mu q^\nu, \quad t_u^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \\ j_q^\mu = q^\mu - (u \cdot q) u^\mu, \quad j_u^\mu = u^\mu q^2 - (u \cdot q) q^\mu,$$

which satisfy relations

$$t_q^{\mu\nu} q_\nu = q_\nu t_q^{\nu\mu} = 0, \quad t_u^{\mu\nu} u_\nu = u_\nu t_u^{\nu\mu} = 0 \\ t_q^{\mu\nu} u_\mu u_\nu = t_u^{\mu\nu} q_\mu q_\nu = q^2 - (u \cdot q)^2, \\ t_{q,\mu}^\mu = 3 q^2, \quad t_{u,\mu}^\mu = 3, \quad j_q^\mu u_\mu = j_u^\mu q_\mu = 0, \\ j_q^\mu q_\mu = j_u^\mu u_\mu = q^2 - (u \cdot q)^2, \\ j_q^\mu j_{u,\mu} = (u \cdot q) [(u \cdot q)^2 - q^2]. \quad (\text{B5})$$

Coefficients in Eqs. (B3) and (B4) are related as

$$A = -C_{qq} - (C_{qu} - C_{uq})(u \cdot q), \\ B = C_{qu} q^2 + C_{uq}(u \cdot q)^2, \\ C = C_{uq} q^2 + C_{qu}(u \cdot q)^2, \\ D = -C_{uu} - (C_{qu} + C_{uq}) q^2 (u \cdot q), \\ E = C_g + C_{uu} + C_{qq} q^2. \quad (\text{B6})$$

Using Eq. (B5) from Eq. (B4) we derive:

$$J_\mu^\mu = 3(q^2 C_{qq} + C_{uu}) + \Omega Q^2 (C_{uq} + C_{qu}) + 4 C_g, \\ J^{\mu\nu} q_\mu q_\nu = J_{qq} = -C_{uu} Q^2 + C_g q^2, \\ J^{\mu\nu} u_\mu q_\nu = J_{uq} = C_{uq} Q^4 + C_g \Omega, \\ J^{\mu\nu} q_\mu u_\nu = J_{qu} = C_{qu} Q^4 + C_g \Omega, \\ J^{\mu\nu} u_\mu u_\nu = J_{uu} = -C_{qq} Q^2 + C_g, \\ Q^2 = (u \cdot q)^2 - q^2, \quad \Omega = (u \cdot q). \quad (\text{B7})$$

Explicit expressions for all C_{ij} -coefficients look clumsy and we do not present them. However, they are easily written through the coefficient C_g , which equals to

$$C_g = \frac{3(J_{qq} + q^2 J_{uu}) + Q^2 J_\mu^\mu - \Omega(J_{qu} + J_{uq})}{2(2\Omega^2 + Q^2)}. \quad (\text{B8})$$

From the definition of the tensor $J^{\mu\nu}$ we can easily find that the various convolutions with the vectors q and u and the trace J_μ^μ can be expressed as

$$J_\mu^\mu = \frac{1}{2}(q^2 - m_1^2) I_0, \quad J_{uu} = \Omega I_1 - I_2, \\ J_{qq} = \frac{1}{2}(q^4 - m_1^4) I_0, \quad J_{uq} = \frac{1}{2}(q^2 - m_1^2) I_1, \\ J_{qu} = \frac{1}{2}(q^2 + m_1^2) [\Omega I_0 - I_1], \quad (\text{B9})$$

where there appeared tree scalar integrals

$$I_n = \int \frac{d^3 q_1}{2\omega_1} \frac{d^3 q_2}{2\omega_2} (u \cdot q_1)^n n(u \cdot q) \delta^{(4)}(q_1 + q_2 - q) \\ = \pi \theta(q^2 - m_1^2) \theta(\omega) \\ \times \int_{\max\{m_1, \omega_-(q)\}}^{\min\{\omega, \omega_+(q)\}} d\omega_1 n(\omega_1) \omega_1^n \sqrt{\omega_1^2 - m_1^2}, \\ \omega_\pm(q) = \frac{\omega(q^2 - m_1^2)}{2q^2} \pm |\vec{q}| \sqrt{\frac{q^4 + m_1^4}{4q^4}}, \quad (\text{B10})$$

$n = 0, 1, 2$. Finally, we can construct the lepton tensor for charged lepton currents

$$T_{\text{lept}}^{\mu\nu} = 8(I^{\mu\nu} + I^{\nu\mu} - g^{\mu\nu} I_\mu^\mu) - 8(J^{\mu\nu} + J^{\nu\mu} - g^{\mu\nu} J_\mu^\mu),$$

where

$$8(J^{\mu\nu} + J^{\nu\mu} - g^{\mu\nu} J_\mu^\mu) = 2C_{qq} t_q^{\mu\nu} + 2C_{uu} t_u^{\mu\nu} \\ + (C_{uq} + C_{qu})(j_u^\mu j_q^\nu + j_q^\mu j_u^\nu) + 2C_g g^{\mu\nu} \\ - (C_{qq} 3q^2 + C_{uu} 3 + (C_{uq} + C_{qu}) \Omega Q^2 + C_g 4) g^{\mu\nu}.$$

For the tensor $\tilde{T}_{\text{lept}}^{\mu\nu}$ we can use the same expression and the relations for the C_{ij} -coefficients with the only replacements $J_\mu^\mu \rightarrow \tilde{J}_\mu^\mu$ and $J_{ab} \rightarrow \tilde{J}_{ab}$, $a, b = q, u$, where

$$\tilde{J}_\mu^\mu = \frac{1}{2}(q^2 - m_1^2) I_1, \quad \tilde{J}_{uu} = \Omega I_2 - I_3, \\ \tilde{J}_{qq} = \frac{1}{2}(q^4 - m_1^4) I_1, \quad \tilde{J}_{uq} = \frac{1}{2}(q^2 - m_1^2) I_2, \\ \tilde{J}_{qu} = \frac{1}{2}(q^2 + m_1^2) [\Omega I_1 - I_2]. \quad (\text{B11})$$

Appendix C: Equilibrium Relations

The equilibrium Kubo-Schwinger-Martin relations between $(-+)$ and $(+-)$ fermion Green's functions and the boson self-energies are

$$G^{-+}(p) = \mp G^{+-}(p) e^{-\varepsilon/T}, \\ \Sigma^{-+}(q) = \pm \Sigma^{+-}(q) e^{-\varepsilon/T}, \quad (\text{C1})$$

see Ref. [145], where $\varepsilon = p_0 - \mu$ with the chemical potential related to the conserved charge. In the case considered in the given paper $\mu \neq 0$ for fermions except for neutrinos/antineutrinos which freely escape from the piece of matter and $\mu = 0$ for bosons. All the Green's functions can be expressed through the retarded and advanced Green's functions

$$G^{--}(p) = G^R(p) \pm i n(\varepsilon) A(p), \quad (\text{C2a})$$

$$G^{++}(p) = -G^A(p) \pm i n_f(\varepsilon) A(p), \quad (\text{C2b})$$

$$G^{-+}(p) = \pm i n(\varepsilon) A(p), \quad (\text{C2c})$$

$$G^{+-}(p) = -i [1 \mp n(\varepsilon)] A(p), \quad (\text{C2d})$$

where $A = 2 \Im G^A = -2 \Im G^R$ is the spectral density and

$$n(\epsilon) = 1/(\exp(\epsilon/T) \pm 1). \quad (\text{C3})$$

Analogously for the self-energies we have

$$\Sigma^{--}(p) = \Sigma^R(p) \pm i n(\epsilon) \Gamma(p), \quad (\text{C4a})$$

$$\Sigma^{++}(p) = -\Sigma^A(p) \pm i n(\epsilon) \Gamma(p), \quad (\text{C4b})$$

$$\Sigma^{-+}(p) = \mp i n(\epsilon) \Gamma(p), \quad (\text{C4c})$$

$$\Sigma^{+-}(p) = i [1 \mp n(\epsilon)] \Gamma(p), \quad (\text{C4d})$$

where $\Gamma = 2 \Im \Sigma^A = -2 \Im \Sigma^R$ is the width.

Now, using (A12) we derive the equilibrium relations among the products of the fermion Green's functions \tilde{L}^{ij} and the functions introduced as $\tilde{L}^R = \tilde{L}^{--} - \tilde{L}^{-+}$ and $\tilde{L}^A = \tilde{L}^{-+} - \tilde{L}^{++} = [\tilde{L}^R]^\dagger$, see Eqs (108,109). Making use of Eq. (C1) we immediately find

$$\begin{aligned} i\tilde{L}^{-+}(p; q) &= G_{p+q/2}^{-+} G_{p-q/2}^{+-} = G_{p+q/2}^{+-} G_{p-q/2}^{-+} e^{-\frac{\epsilon}{T}} \\ &= i\tilde{L}^{+-}(p; q) e^{-\frac{\epsilon}{T}}. \end{aligned} \quad (\text{C5})$$

From Eq. (110) follows

$$\begin{aligned} \mathcal{G}_0 \cdot (\tilde{L}^{-+} - \tilde{L}^{+-}) \cdot \mathcal{G}_0 &= \mathcal{G}_0 \cdot (\tilde{L}^A - \tilde{L}^R) \cdot \mathcal{G}_0 \\ &= -2i \Im(\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0). \end{aligned} \quad (\text{C6})$$

Combining Eqs. (C5) and (C6) we obtain

$$\mathcal{G}_0 \cdot \tilde{L}^{-+} \cdot \mathcal{G}_0 = \frac{2i \Im(\mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0)}{e^{\frac{\epsilon}{T}} - 1} \quad (\text{C7})$$

and recover Eq. (121a). Equation (121b) follows then from Eqs. (C7) and (C5). From the definition of the retarded function and Eq. (110) we have

$$\begin{aligned} \mathcal{G}_0 \cdot \tilde{L}^{--} \cdot \mathcal{G}_0 &= \mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0 + \mathcal{G}_0 \cdot \tilde{L}^{-+} \cdot \mathcal{G}_0, \\ \mathcal{G}_0 \cdot \tilde{L}^{++} \cdot \mathcal{G}_0 &= \mathcal{G}_0 \cdot \tilde{L}^{+-} \cdot \mathcal{G}_0 - \mathcal{G}_0 \cdot \tilde{L}^R \cdot \mathcal{G}_0, \end{aligned} \quad (\text{C8})$$

which together with Eqs. (C7) and (C5) translate into Eqs. (121c) and (121d).

For completeness, we list also the fermionic quasi-particle Green's functions for a system with paring in equilibrium

$$\begin{aligned} \hat{G}^{-+}(\epsilon, \vec{p}) &= 2\pi i a n_f(\epsilon) \\ &\quad \times [u_p^2 \delta(\epsilon - E_p) + v_p^2 \delta(\epsilon + E_p)] \sigma_0, \\ \hat{F}^{(1,2)-+}(\epsilon, \vec{p}) &= 2\pi i a n_f(\epsilon) \frac{\hat{\Delta}^{(1,2)}(p)}{4 E_p^2} \\ &\quad \times [\delta(\epsilon + E_p) - \delta(\epsilon - E_p)], \\ \hat{G}^{+-}(\epsilon, \vec{p}) &= -2\pi i a [1 - n_f(\epsilon)] \\ &\quad \times [u_p^2 \delta(\epsilon - E_p) + v_p^2 \delta(\epsilon + E_p)] \sigma_0, \\ \hat{F}^{(1,2)+-}(\epsilon, \vec{p}) &= 2\pi i a [1 - n_f(\epsilon)] \frac{\hat{\Delta}^{(1,2)}(p)}{4 E_p^2} \\ &\quad \times [\delta(\epsilon - E_p) - \delta(\epsilon + E_p)]. \end{aligned}$$

Here the quasi-particle spectrum is given by $E_p^2 = \epsilon_p^2 + \Delta^2(p, T)$ with

$$\Delta^2(p) = -a^2 \frac{1}{2} \text{Tr}\{\hat{\Delta}^{(1)}(p) \hat{\Delta}^{(2)}(p)\}$$

and $\epsilon_p = (p^2 - p_F^2)/(2 m_N^*)$. The Bogolyubov's factors are

$$u_p^2 = \frac{E_p + \epsilon_p}{2E_p}, \quad v_p^2 = \frac{E_p - \epsilon_p}{2E_p}. \quad (\text{C9})$$

For the Green's functions for holes

$$\hat{G}^{h\pm\mp}(p) = \sigma_2 \hat{G}^{\pm\mp}(-p) \sigma_2.$$

Expressions for normal systems are obtained from here for $v_p \rightarrow 0$.

Appendix D: The loop functions

At zero temperature the loop functions (199) were calculated in Ref. [143] using the Feynman method for the integral of the Green's function products

$$\begin{aligned} L(\vec{n}, q; P) &= a^2 \rho \left[\frac{\vec{q} \vec{v}}{\omega - \vec{q} \vec{v}} (1 - g(z)) - \frac{g(z)}{2} (1 + P) \right], \\ M(\vec{n}, q) &= -a^2 \rho \frac{\omega + \vec{q} \vec{v}}{2 \Delta} g(z), \\ N(\vec{n}, q) &= a^2 \rho \frac{\omega^2 - (\vec{q} \vec{v})^2}{4 \Delta^2} g(z), \\ O(\vec{n}, q; P) &= a^2 \rho \left[\frac{\omega + \vec{q} \vec{v}}{4 \Delta} + \frac{\omega - \vec{q} \vec{v}}{4 \Delta} P \right] g(z). \end{aligned} \quad (\text{D1})$$

Here the variable P can be an operator as in Eq. (199) or (± 1) value as in Eq. (210). The universal function g is given by

$$g = 2 \int d\Phi_0 \frac{F_+ F_-}{a^2 \rho} = \int \frac{2 \Delta^2 d\Phi_0}{[\epsilon_+^2 - E_+^2][\epsilon_-^2 - E_-^2]},$$

where $\int d\Phi_0$ is $\int d\Phi_T$ at $T = 0$, see Eq. (192); $\epsilon_{\pm} = \epsilon \pm \frac{1}{2}\omega$ and $E_{\pm} = E_{p \pm q/2}$. Evaluation of the integral yields

$$\begin{aligned} g(z) &= -\frac{\text{arcsinh} \sqrt{z^2 - 1}}{z \sqrt{z^2 - 1}} - \frac{i \pi \theta(z^2 - 1)}{2 z \sqrt{z^2 - 1}}, \quad (\text{D2}) \\ z^2 &= \frac{\omega^2 - (\vec{q} \vec{v})^2}{4 \Delta^2} > 1, \quad \vec{v} = v_F \vec{n}. \end{aligned}$$

In various limiting cases one obtains

$$\begin{aligned} g(z) &\simeq 1 - z^2/2 \quad \text{for } |z| \ll 1, \\ g(z) &\rightarrow -i \frac{\pi}{2|z| \sqrt{-1 - z^2}} \quad \text{for } z^2 \rightarrow -1, \\ g(z) &\rightarrow \frac{\ln(2z)}{z^2} \quad \text{for } z^2 \rightarrow \infty, \\ g(z) &\rightarrow -\frac{\ln(2|z|)}{|z^2|} - i \frac{\pi}{2|z^2|} \quad \text{for } -z^2 \rightarrow \infty, \end{aligned} \quad (\text{D3})$$

and for

$$\frac{(\vec{q}\vec{v}_F)^2}{8\Delta} < |\omega - 2\Delta| \ll 2\Delta$$

we have

$$g(z) \simeq -i \frac{\pi\sqrt{\Delta}}{2|z|\sqrt{\omega-2\Delta}} \left(1 + \frac{(\vec{q}\vec{v}_F)^2}{8\Delta(\omega-2\Delta)}\right) - 1 + O\left[\left(\frac{(\vec{q}\vec{v}_F)^2}{\Delta(\omega-2\Delta)}\right)^2\right]. \quad (\text{D4})$$

At finite temperatures the Feynman method does not work [146] and the Matsubara technique can be used instead. In Ref. [95] it was shown that the functions L , M , N , and O can be expressed through one universal temperature-dependent function

$$g_T(\vec{n}, i\omega_m, \vec{q}) = 2 \int d\Phi_T \frac{F_+ F_-}{a^2 \rho} = 2 \Delta^2 \quad (\text{D5})$$

$$\times \int_{-\infty}^{+\infty} d\epsilon_p \sum_{n=-\infty}^{+\infty} \frac{T}{[(i\epsilon_n + i\omega_m)^2 - E_+^2][(i\epsilon_n)^2 - E_-^2]}.$$

Here $\epsilon_n = (2n+1)\pi T$ and $\omega_m = 2m\pi T$. After the summation over n we obtain

$$g_T(\vec{n}, i\omega_m, \vec{q}) = \Delta^2 \int_{-\infty}^{+\infty} d\epsilon_p \times \left[\frac{(E_+ - E_-)}{E_+ E_-} \frac{(n_f(E_-) - n_f(E_+))}{(i\omega_m)^2 - (E_+ - E_-)^2} - \frac{(E_+ + E_-)}{E_+ E_-} \frac{(1 - n_F(E_-) - n_f(E_+))}{(i\omega_m)^2 - (E_+ + E_-)^2} \right]. \quad (\text{D6})$$

After the replacement $i\omega_m \rightarrow \omega_+ = \omega + i0$ we obtain the analytical continuation to the retarded g_T function in the ω -complex plane. The expressions for the loop functions read

$$M(\vec{n}, \omega, \vec{q}) = -a^2 \rho \frac{\omega + \vec{q}\vec{v}}{2\Delta} g_T(\vec{n}, \omega, \vec{q}),$$

$$N(\vec{n}, \omega, \vec{q}) = a^2 \rho \frac{\omega^2 - (\vec{q}\vec{v})^2}{4\Delta^2} g_T(\vec{n}, \omega, \vec{q}),$$

$$O(\vec{n}, \omega, \vec{q}; P) = a^2 \rho \left[\frac{\omega + \vec{q}\vec{v}}{4\Delta} + \frac{\omega - \vec{q}\vec{v}}{4\Delta} P \right] \times g_T(\vec{n}, \omega, \vec{q}),$$

$$L(\vec{n}, \omega, \vec{q}; P) = a^2 \rho \left[\frac{\vec{v}\vec{q}}{\omega - \vec{v}\vec{q}} (g_T(\vec{n}, (\vec{v}\vec{q}), \vec{q}) - g_T(\vec{n}, \omega, \vec{q})) - \frac{1+P}{2} g_T(\vec{n}, \omega, \vec{q}) \right]. \quad (\text{D7})$$

For $T = 0$, $g_T(\vec{n}, (\vec{v}\vec{q}), \vec{q}) = 1$ and the old result (D1) is recovered.

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